1 Topic 3: Electrostatic Fields in Matter

Reading Assignment: Jackson Chapter 4.3 - 4.7

1.1 Polarization

If a neutral atom is place in an electric field, it can become polarized, where the induced dipole moment $\mathbf{p}$ is approximately proportional to the field, as long as its not too strong

$$\mathbf{p} = \alpha \mathbf{E} \tag{1}$$

The constant of proportionality is called the *atomic polarizability*. For molecules, the situation isn't as simple, because they polarize more readily in some directions that others. In this case the induced dipole moment may not even be in the same direction as $\mathbf{E}$, and the most general relationship is

$$\mathbf{p} = \overline{\boldsymbol{\alpha}} \cdot \mathbf{E} \tag{2}$$

i.e.

$$p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z \tag{3}$$

where α_{xx} are the elements of the *polarizability tensor*.

Some molecules have built-in, permanent dipole moments (e.g. water has a big one – $6.1 \times 10^{-30} C \cdot m$, which is why its an effective solvent). In an electric field, the atomic dipoles will experience a torque

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \tag{4}$$

that lines $\mathbf{p}$ up parallel to $\mathbf{E}$. If the field is nonuniform, there will also be a net force on the dipole,

$$\mathbf{F} = \mathbf{F}_{+chg} + \mathbf{F}_{-chg} = q\left(\mathbf{E}_{+} - \mathbf{E}_{-}\right) = q\left(d\mathbf{E}\right) \tag{5}$$

where $d\mathbf{E}$ is the difference between the field at the $+$ and $-$ ends of the dipole. If the dipole is very short,

$$dE_x = \left(\boldsymbol{\nabla} E_x\right) \cdot \mathbf{s} \tag{6}$$

and similarly for the other components, so

$$d\mathbf{E} = \left(\mathbf{s} \cdot \boldsymbol{\nabla}\right) \mathbf{E} \tag{7}$$

and

$$\mathbf{F} = \left(\mathbf{p} \cdot \boldsymbol{\nabla}\right) \mathbf{E} \tag{8}$$

This is a useful formula, but remember it only applies to perfect dipoles of infinitessimal length.

Whatever the mechanism (alignment of permanent dipole moments or induced polarization of the atoms) the net result is the same – a lot of little dipoles pointing along the direction of the field, and the dielectric becomes polarized. The *polarization*

$$\mathbf{P} \equiv \text{dipole moment per unit volume} \tag{9}$$

1.2 The Field of a Polarized Object

If we have some polarized material with polarization $\mathbf{P}$, what is the field produced by this object? (note – not the field which may have caused the polarization) For a single dipole

$$\Phi = \frac{1}{4\pi\varepsilon_o}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} \tag{10}$$

where $\mathbf{r}$ is the vector from the dipole to the point at which we are evaluating the potential. Integrating over the volume containing the dielectric, considering we have a dipole moment in each unit volume $\mathbf{p} = \mathbf{P}d\tau$

$$\begin{aligned}\Phi &= \frac{1}{4\pi\varepsilon_o}\int_V \frac{\mathbf{P}\cdot\mathbf{r}}{r^2}d\tau \\ &= \frac{1}{4\pi\varepsilon_o}\int_V \mathbf{P}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right)d\tau\end{aligned}$$

or integrating by parts

$$\Phi = \frac{1}{4\pi\varepsilon_o}\left[\int_V \boldsymbol{\nabla}\cdot\left(\frac{1}{r}\mathbf{P}\right)d\tau - \int_V \frac{1}{r}\left(\boldsymbol{\nabla}\cdot\mathbf{P}\right)d\tau\right] \tag{11}$$

or using the divergence theorem

$$\Phi = \frac{1}{4\pi\varepsilon_o}\left[\int_S \frac{1}{r}\mathbf{P}\cdot\mathbf{n}\,da - \int_V \frac{1}{r}\left(\boldsymbol{\nabla}\cdot\mathbf{P}\right)d\tau\right] \tag{12}$$

The first term looks just like th epotential of a surface charge

$$\sigma_b = \mathbf{P}\cdot\mathbf{n} \tag{13}$$

while the second looks like the potential from a volume charge

$$\rho_b = -\nabla\cdot\mathbf{P} \tag{14}$$

So we can write

$$\Phi = \frac{1}{4\pi\varepsilon_o}\left[\int_S \frac{1}{r}\sigma_b\,da - \int_V \frac{1}{r}\left(\rho_b\right)d\tau\right] \tag{15}$$

So the potential (and also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\boldsymbol{\nabla}\cdot\mathbf{P}$ plus a surface charge density $\sigma_b = \mathbf{P}\cdot\mathbf{n}$. So, instead of integrating over all the infinitesimal dipoles, we find the bound charghes they produce, and then calculate the fields form these bound charges.

Note that these bound charges are perfectly genuine accumulations of charge, not "fictitious". If you have a string of dipoles you'll end up with a + at one end and a - at the other. These corresond to the surface charge - as if we had peeled an electron off one end and put it at the other.

Note that for any uniform polarization, we only end up with bound surface charges. We need a non-uniform polarization - i.e. a divergence of the polarization field - to get a bound volume charge.

1.2.1 The Field Inside a Dielectric

We've been careless about the distinction between pure dipoles and physical dipoles in the above – we assumed that we were working with pure dipoles. Our polarized molecules or atoms are physical dipoles. Further, we represented discrete molecular dipoles by a continuous density function **P**. Outside the dielectric this is clearly not a problem, since we are far enough away that the dipole potential dominates, and the "graininess" of the medium doesn't matter. Inside, however, this is clearly not the case.

The actual field inside is clearly incredibly complicated, and the microscopic field would be impossible to calculate. We're really interested in the macroscopic field – the average field over regions large enough that they contain many thousands of atoms.

How do we know that we are getting the average, macroscopic field inside the dielectric when we use the procedure of bound charges? The argument is subtle. If we want the macroscopie field at some point P inside the dielectric, we must average the true, microscopic field over a volume large enough that it contains many atoms. Consider a sphere of radius many atomic diameters. The macroscopic field at P consists of the average field over the sphere due to all charges outside plus the average due to all charges inside

$$\mathbf{E} = \mathbf{E}_{out} + \mathbf{E}_{in} \tag{16}$$

The average field produced by charges outside a sphere is equal to the field they produce at the center - this isn't hard to prove. So $\mathbf{E}_{out}$ is the field at P due to the dipoles exterior to the sphere, which are far enough away that

$$\Phi_{out} = \frac{1}{4\pi\varepsilon_o} \int_{outside} \frac{\mathbf{P}\cdot\mathbf{r}}{r^3}\, d\tau \tag{17}$$

The dipoles inside the sphere are too close to treat in this fashion. You showed in your homework that the average field inside a sphere for any charge distribution contained in the sphere is

$$\mathbf{E}_{in} = -\frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}}{R^3} \tag{18}$$

regardless of the details of the charge distribution within the sphere. The only important quantity is the total dipole moment $\mathbf{p} = \left(\frac{4}{3}\pi R^3\right)\mathbf{P}$

$$\mathbf{E}_{in} = -\frac{1}{3\varepsilon_o}\mathbf{P} \tag{19}$$

Now we have the curious fact that the average field over any sphere (due to the charge inside) is the same as the field at the center of a uniformly polarized sphere with the same total dipole moment $\mathbf{E} = -\frac{1}{3\varepsilon_o}\mathbf{P}$. This means that if we want the total potential at the center of the sphere,

$$\Phi = \frac{1}{4\pi\varepsilon_o} \int \frac{\mathbf{P}\cdot\mathbf{r}}{r^3} d\tau \tag{20}$$

since no matter how crazy the actual microscopic charge configuration is, we can replace it by a smooth distribution of perfect dipoles. So, we inadvertantly did the right thing before.

1.3 The Electric Displacement

We want to look at how to modify Gauss' law when we have dielectric media. The field due to polarization of the media is the field due to the bound charge distributions. We may also have free charge – anything not due to polarization

$$\rho = \rho_b + \rho_f \tag{21}$$

and Gauss' law is

$$\varepsilon_o \nabla \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\nabla \cdot \mathbf{P} + \rho_f \tag{22}$$

where $\mathbf{E}$ is the total field. We can rearrange this

$$\nabla \cdot (\varepsilon_o \mathbf{E} + \mathbf{P}) = \rho_f \tag{23}$$

and define

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} \tag{24}$$

where $\mathbf{D}$ is called the *electric displacement*. We get

$$\nabla \cdot \mathbf{D} = \rho_f \tag{25}$$

This is convenient, as it makes reference only to free charges, which is what we generally control.

A caution. You should not get the idea that $\mathbf{D}$ is just like $\mathbf{E}$ except that its source is ρ_f instead of ρ. There is no Coulomb's law for $\mathbf{D}$

$$D \neq \frac{1}{4\pi} \int \rho_f \left(\frac{\mathbf{r}}{\mathbf{r}^3} \right) d\tau \tag{26}$$

The parallel between $\mathbf{D}$ and $\mathbf{E}$ is more subtle.

The reason is the divergence alone is insufficient to determine a vector field – you need the curl as well. We always (for electrostatics) have

$$\nabla \times \mathbf{E} = 0 \tag{27}$$

but

$$\nabla \times \mathbf{D} \neq 0 \tag{28}$$

(consider the bar electret). $\mathbf{D}$ and $\mathbf{E}$ have different boundary conditions: the tangential component of $\mathbf{E}$ is continuous accross a boundary, but the tangential component of $\mathbf{D}$ is not. The normal component of $\mathbf{D}$ changes by σ_f.

1.4 Linear Dielectrics

For linear dielectrics

$$\mathbf{P} = \varepsilon_o \chi_e \mathbf{E} \tag{29}$$

where χ_e is called the electric ssusceptibility of the medium. Here E is the total field (due to free charge and polarizaiton).

In linear media

$$\mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P} = \varepsilon_o \mathbf{E} + \varepsilon_o \chi_e \mathbf{E} = \varepsilon_o (1 + \chi_e) \mathbf{E} \tag{30}$$

Thus both **D** and **P** are proportional to **E**.

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E} \\ \varepsilon &= \varepsilon_o (1 + \chi_e) \end{aligned}$$

ε is called the permittivity of the material. The dielectric constant is defined as the dimensionless

$$K = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_o} \tag{31}$$

In a linear medium

$$\rho_b = -\nabla \cdot \mathbf{P} = -\nabla \cdot \left(\varepsilon_o \frac{\chi_e}{\varepsilon} \mathbf{D} \right) = -\left(\frac{\chi_e}{1 + \chi_e} \right) \rho_f \tag{32}$$

Unless there is free charge embedded in the medium, all the bound charge resides on the surface.

If all space is filled with a dielectric (so we don't have to worry about any boundaries) then $\nabla \times \mathbf{D} = 0$, and $\nabla \cdot \mathbf{D} = \rho_f$, and everwhere **D** can be found from the free charge only

$$\mathbf{D} = \varepsilon_o \mathbf{E}_{vac} \tag{33}$$

and

$$\mathbf{E} = \frac{1}{K} \mathbf{E}_{vac} \tag{34}$$

the electric field is reduced from its vacuum value by the presence of the dielectric.

1.5 Energy in a Dielectric System

Lets consider the energy stored in a dielectric system the following way Suppose the dielectric system is fixed in position and we bring in the free charge a bit at a time. As ρ_f is increased by an amount $\Delta\rho_f$ the polarization will change as will the bound charge distribution. But we're interested only in the work ond on the incremental free charge

$$\Delta W = \int \left(\Delta \rho_f \right) \Phi \, d\tau \tag{35}$$

now $\nabla \cdot \mathbf{D} = \rho_f$, so $\Delta\rho_f = \nabla \cdot (\Delta \mathbf{D})$ and

$$\Delta W = \int \left(\nabla \cdot (\Delta \mathbf{D}) \right) \Phi \, d\tau \tag{36}$$

Now

$$\nabla \cdot (\Delta \mathbf{D} \boldsymbol{\Phi}) = \left[\nabla \cdot (\Delta \mathbf{D}) \right] \Phi + \Delta \mathbf{D} \cdot (\nabla \Phi) \tag{37}$$

so

$$\Delta W = \int \nabla \cdot (\Delta \mathbf{D} \Phi) \; d\tau + \int (\Delta \mathbf{D}) \cdot \mathbf{E} \, d\tau \tag{38}$$

we can get rid of the first term by turning it into a surface interval using the divergence theorem and integrating over all space. Therefore

$$\Delta W = \int (\Delta \mathbf{D}) \cdot \mathbf{E}\, d\tau \tag{39}$$

The above applies to any material. If we have a linear dielectric then $\mathbf{D} = \varepsilon \mathbf{E}$ and

$$\frac{1}{2}\Delta\left(\mathbf{D}\cdot\mathbf{E}\right) = \frac{1}{2}\Delta(\varepsilon E^2) = \varepsilon\left(\Delta\mathbf{E}\right)\cdot\mathbf{E} = (\Delta\mathbf{D})\cdot\mathbf{E} \tag{40}$$

Thus

$$\Delta W = \Delta\left(\int \frac{1}{2}\mathbf{D}\cdot\mathbf{E}\, d\tau\right) \tag{41}$$

The total work done as we build the charge configuation is then

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\, d\tau \tag{42}$$

Now we derived $W = \frac{1}{2}\varepsilon_o \int E^2 d\tau$ quite generally a few months ago. Why doesn't it apply here? It refers to a different energy – the work required to assemble the free and bound charge, but it doesn't include the energy stored in the dielectric due to polarizing (stretching and twisting) the molecules. The latter expression does - since it refers to the energy reqired to assemble the free charge allowing the dielectric to respond as it sees fit - so it indirectly includes the "spring energy"

$$W_{tot} = W_{free} + W_{bound} + W_{spring} \tag{43}$$

The last two are equal and opposite since the dielectric is always in equilibrium (net work on bound charge is zero) so by calculating W_{free} we got W_{tot}.

1.5.1 Forces on Dielectrics

Consider a dielectric being inserted between the plates of a capacitor. Let the width of the plates be w and the distance inserted be s. There will be a force pulling the dielectric into the capacitor (where does the force come from?).

We can calculate the force from the change in energy

$$dW = F_a\, ds \tag{44}$$

where dW is the change in energy if we pull the dielectric out an infinitessimal amount, and F_a is the force we apply. The electrical force on the slab is

$$F = -\frac{dW}{ds} \tag{45}$$

Now

$$W = \frac{1}{2}CV^2 \tag{46}$$

where V is the voltage on the capacitor, and

$$C = \frac{\varepsilon_o a}{d}\left(w + \chi_e s\right) \tag{47}$$

As the dielectric moves, the potential changes – what stays constant is the total charge on the plates

$$W = \frac{1}{2}\frac{Q^2}{C} \tag{48}$$

so that

$$F = -\frac{dW}{ds} = \frac{1}{2}\frac{Q^2}{C^2}\frac{dC}{ds} = \frac{1}{2}V^2\frac{dC}{ds} \tag{49}$$

but

$$\frac{dC}{ds} = \frac{\varepsilon_o \chi_e a}{d} \tag{50}$$

so

$$F = \frac{1}{2}\varepsilon_o \chi_e \frac{a}{d} V^2 \tag{51}$$

(think about how you would solve this same problem given V constant.)

1.5.2 Polarizability and Susceptibility

For a linear dielectric

$$\mathbf{P} = \varepsilon_o \chi_e \mathbf{E} \tag{52}$$

If the material consists of non-polar atoms or molecules, then for each

$$\mathbf{p} = \alpha \mathbf{E} \tag{53}$$

What is the connection between α and χ_e?

The number of atoms/unit volume is N, and one would naievely guess

$$\chi_e = \frac{N\alpha}{\varepsilon_o} \tag{54}$$

There's a subtle problem, however. In

$$\mathbf{P} = \varepsilon_o \chi_e \mathbf{E} \tag{55}$$

$\mathbf{E}$ is the total microscopic field due to everything except the particular atom in question.

The microscopic field is impossible to calculate exactly, but we get a pretty good estimate from the following. Assume each atom occupies a volume $4/3\pi R^3$ so that $N = 1/(4/3\pi R^3)$. The macroscopic field

$$\mathbf{E} = \mathbf{E}_{self} + \mathbf{E}_{else} \tag{56}$$

where $\mathbf{E}_{self}$ is the average over the sphere due to the atom itself (equal to the value at the center) and $\mathbf{E}_{else}$ is the average inside due to everything outside. Then

$$\begin{aligned} \mathbf{p} &= \alpha \mathbf{E}_{else} \\ \mathbf{P} &= N\alpha \mathbf{E}_{else} \end{aligned}$$

Now

$$\mathbf{E}_{self} = -\frac{1}{4\pi\varepsilon_o}\frac{\mathbf{p}}{R^3} \tag{57}$$

and

$$\begin{aligned} \mathbf{E} &= \frac{-1}{4\pi\varepsilon_o}\frac{\alpha}{R^3}\mathbf{E}_{else} + \mathbf{E}_{else} \\ &= \left(1 - \frac{\alpha}{4\pi\varepsilon_o R^3}\right)\mathbf{E}_{else} \end{aligned}$$

or

$$\mathbf{E} = \left(1 - \frac{N\alpha}{3\varepsilon_o}\right)\mathbf{E}_{else} \tag{58}$$

and

$$\mathbf{P} = \frac{N\alpha}{(1 - N\alpha/3\varepsilon_o)}\mathbf{E} \tag{59}$$

so

$$\chi_e = \frac{N\alpha}{(1 - N\alpha/3\varepsilon_o)} \tag{60}$$

This is somewhat remarkable if you think about it. The macroscopic quantity χ_e is so simply related to the atomic parameter α. Usually this is written in terms of the dielectric constant

$$\alpha = \frac{3\varepsilon_o}{N}\left(\frac{K-1}{K+2}\right) \tag{61}$$

and is called the Clausius-Mossotti formula.

To go further and consider polar molecules we need statistical mechanics.