Midterm: Mean 14.4/30, sigma = 5, high score - 25/30

1 Topic 3: Magnetostatic Fields in Matter

Reading Assignment: Jackson Chapter 5.7 - 5.11

The treatment of magnetostatic fields in matter is quite parallel to our discussion of dielectrics. There are, however, a number of important differences, so some caution is advised in drawing blind parallels.

1.1 Magnetization

With dielectrics, the polarization is almost always in the same direction as $\mathbf{E}$, but materials can acquire a magnetization either parallel to $\mathbf{B}$ (paramagnets), or antiparallel to $\mathbf{B}$ (diamagnets). Materials can also be permanently magnetized (ferromagnets), but we'll deal with that later.

1.1.1 Torques and Forces on Magnetic Dipoles

The basic mechanisms for magnetization of materials are the alignment of electron spin magnetic moments in the direction of the magnetic field (paramagnetism), and the acceleration of the electrons in their orbits (diamagnetism). Serious treatment of both of these really requires quantum mechanics, but we can get an idea for the basic physics with a simple classical treatment.

First, let's look at the torque and force on a magnetic dipole in a $\mathbf{B}$-field. Consider a current loop with its center at the origin in a uniform field pointing in the z-direction. Let the normal to the loop be tilted at angle θ with respect to the z-axis. The forces on the two sides parallel to the y-axis (the tilted sides) stretch the loop, but don't twist it (see picture), whereas the forces on the sides parallel to the x-axis result in a torque tending to align the plane of the loop perpendicular to the B-field. The torque is

$$\mathbf{N} = aF\sin\theta\,\hat{\mathbf{e}}_x \tag{1}$$

The magnitude of the force on each segment is

$$F = IbB \tag{2}$$

and

$$\mathbf{N} = (Iab)\,B\sin\theta\,\hat{\mathbf{e}}_x = mB\sin\theta\,\hat{\mathbf{e}}_x = \mathbf{m}\times\mathbf{B} \tag{3}$$

where $m = Iab$ is the magnetic dipole moment of the loop. This is the exact torque in a uniform field, or for a point dipole in a nonuniform field.

This torque accounts for paramagnetism, where each the electron dipole moments are acted on by the field. Paramegnetism normally occurs where the atom or molecules have an odd number of electrons, since the Pauli principle dictates that spins will occur in antiparallel pairs.

Let's look at the force on a current loop. In a uniform field it is zero:

$$\mathbf{F} = I \oint (dl \times \mathbf{B}) = I \left(\oint dl \right) \times \mathbf{B} = 0 \tag{4}$$

since B is constant and comes outside the integarl. In a nonuniform field, for an infinitesimal loop of dipole moment $\mathbf{m}$ in field $\mathbf{B}$ the force is

$$\mathbf{F} = \nabla (\mathbf{m} \cdot \mathbf{B}) \tag{5}$$

Using the product rule

$$\mathbf{F} = \mathbf{m} \times (\boldsymbol{\nabla} \times \mathbf{B}) + (\mathbf{m} \cdot \boldsymbol{\nabla}) \mathbf{B} \tag{6}$$

Here B is the external field, and if there is no external current at the location of the dipole, $\boldsymbol{\nabla} \times \mathbf{B} = \mu_o \mathbf{J}$ implies

$$\mathbf{F} = (\mathbf{m} \cdot \boldsymbol{\nabla}) \mathbf{B} \tag{7}$$

This will remind you of the electrical analog, $\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}$. This is what lead physicists to think of dipoles as consisting of magnetic charges - but note this is of course incorrect, and the distinction between magnetic charges and currents is evident close to the dipole.

1.1.2 Simple Classical Model for Diamagnetism

Although requiring a quantum mechanical treatment, we can understand the basic physics of diamagnetism again with a classical model. Electrons have spin (which accounts for paramagnetism) but they also orbit around the nucleus. Although not technically a steady current, if we think of a circular orbit of radius r with period $T = 2\pi r/v$ that is very short, then

$$I = \frac{e}{T} = \frac{ev}{2\pi r} \tag{8}$$

and the orbital dipole moment is

$$\mathbf{m} = -\frac{1}{2} evr \ \hat{\mathbf{e}}_z \tag{9}$$

Like any other magnetic dipole, this is subject to a torque when the atom is placed in a magnetic field. Since it is much harder to distort the orbit than tilt the spin the orbital contribution to paramagnetism is small. A more significant effect is the change in velocity of the electron which depends on the orientation of $\mathbf{B}$. Without a B-field the $\mathbf{F} = m\mathbf{a}$ implies

$$\frac{1}{4\pi\varepsilon_o} \frac{e^2}{r^2} = m_e \frac{v^2}{r} \tag{10}$$

With a magnetic field there's the additional term $-e(\mathbf{v} \times \mathbf{B})$. If B is perpendicular to the orbital plane, then

$$\frac{1}{4\pi\varepsilon_o} \frac{e^2}{r^2} + ev'B = m_e \frac{v'^2}{r} \tag{11}$$

and the new speed, v' is greater than v

$$ev'B = \frac{m_e}{r} \left(v'^2 - v^2 \right) = \frac{m_e}{r} (v' + v)(v' - v) \tag{12}$$

or if $\Delta v = v' - v$ is small

$$\Delta v = \frac{erB}{2m_e} \tag{13}$$

Note that we have assumed r remains constant - in other words the orbit is fixed. We can't find general justification for this, but if we assume v is constant and r changes our result will only be different by a factor 2. We are looking for a qualitative understanding - anything quantitative requires quantum mechanics.

A change in orbital speed changes the dipole moment *in the opposite direction* as B

$$\Delta \mathbf{m} = \frac{-1}{2} e \left(\Delta v\right) r \,\hat{\mathbf{e}}_z \tag{14}$$

An electron orbiting in the opposite sense would have a dipole moment pointing upward, but this orbit would be slowed down by the field, so the change is still opposite B. In unmagnetized material the orbital dipole moments are randomly oriented, and so cancel out. In a magnetic field, the additional components are all antiparallel to the field, which is the mechanism responsible for diamagnetism. This effect is generally much weaker than paramagnetism, and therefore only observed in atoms with even numbers of electrons, where paramagnetism is absent.

1.1.3 Magnetization

We've looked at two mechanisms by which material in a magnetic field becomes magnetized. We can develop a model similar to what we did for dielectrics, where we define

$$\mathbf{M} = \text{ magnetic dipole moment per unit volume} \tag{15}$$

where $\mathbf{M}$ is called the magnetization. We don't have to be concerned about whether it is diamagnetism or paramagnetism for now.

As an aside - diamagnetism and paramagnetism are very weak phenomena in general. It takes sensitive experiments to detect them. If you suspend a piece of paramagnetic material above a solenoid, the induced magnetization would be upward, and the force downward. Its the opposite for diamagnetic materials. In the same arrangement the force on a piece of iron (nickel or cobalt) would be $10^4 - 10^5$ greater. That is why it is reasonable to calculate the field inside a piece of copper wire without worrying about the effects of magnetization.

1.1.4 Bound Currents

What field does a piece of magnetized material with magnetization $\mathbf{M}$ produce? The vector potential of a single dipole is

$$\mathbf{A} = \frac{\mu_o}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{e}}_r}{r^2} \tag{16}$$

If we have a magnetized object, then

$$\begin{aligned} \mathbf{A} &= \frac{\mu_o}{4\pi} \int \frac{\mathbf{M} \times \hat{\mathbf{e}}_r}{r^2} d\tau \\ &= \frac{\mu_o}{4\pi} \int \left(\mathbf{M} \times \boldsymbol{\nabla} \frac{1}{r} \right) d\tau \end{aligned}$$

Integrating by parts

$$\mathbf{A} = \frac{\mu_o}{4\pi}\left[\int \frac{1}{r}\left(\boldsymbol{\nabla}\times\mathbf{M}\right)d\tau - \int \nabla\times\left(\frac{1}{r}\mathbf{M}\right)d\tau\right] \tag{17}$$

and using a corollary of the curl theorem

$$\mathbf{A} = \frac{\mu_o}{4\pi}\left[\int \frac{1}{r}\left(\boldsymbol{\nabla}\times\mathbf{M}\right)d\tau - \oint_S \frac{1}{r}\left(\mathbf{M}\times d\mathbf{a}\right)\right] \tag{18}$$

The first term looks just like the potential of a volume current

$$\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M} \tag{19}$$

and the second looks like a surface current

$$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} \tag{20}$$

so we can write

$$\mathbf{A} = \frac{\mu_o}{4\pi}\int_V \frac{1}{r}\mathbf{J}_b d\tau + \frac{\mu_o}{4\pi}\oint_S \frac{1}{r}\mathbf{K}_b da \tag{21}$$

So the field of a magnetized object is the same as what would be produced by the volume current $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M}$ and the surface current $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$ on the boundary.

Note that these currents are "real" in the sense they arise from actual currents flowing at an atomic or molecular level, but are distinct from free currents in that charge does not flow freely on the surface, but remains localized. The bound volume current arises only if we have a nonuniform magnetization. For a uniform magnetization the internal currents cancel (think of many rectangular loops with current circulating in the same direction), but if we have two adjacent chunks of magnetized material with greater magnetization in one (say the one at $y + dy$), then we get a net current flowing in the x-direction

$$I_x = \left[M_z\left(y+dy\right) - M_z\left(y\right)\right]dz = \frac{\partial M_z}{\partial y}dy\,dz \tag{22}$$

or the corresponding volume current

$$\left(J_b\right)_x = \frac{\partial M_z}{\partial y} \tag{23}$$

Likewise for the other directions, so we have $\mathbf{J}_b = \boldsymbol{\nabla}\times\mathbf{M}$.

Note that we can make the same arguments relating the microscopic magnetic field inside the material to the macroscopic field that we made for polarization, if we average over regions large enough to contain many atoms.

1.1.5 The Auxiliary Field H

We can write the total current as

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f \tag{24}$$

This separation of bound and free currents is just a convenience - there's no new physics here. We can now write Ampere's law

$$\frac{1}{\mu_o}(\boldsymbol{\nabla} \times \mathbf{B}) = \mathbf{J} = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J}_f + (\boldsymbol{\nabla} \times \mathbf{M}) \tag{25}$$

so

$$\nabla \times \left(\frac{1}{\mu_o}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f \tag{26}$$

We define

$$\mathbf{H} \equiv \frac{1}{\mu_o}\mathbf{B} - \mathbf{M} \tag{27}$$

so Ampere's law written in terms of **H** becomes

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f \tag{28}$$

So **H** plays a role analogous to **D** in electrostatics. **H** is, however, a more useful quantity than **D**, actually for practical reasons. In the lab you control currents and voltages. To make an electromagnet you run a known free current, which determines (the line integral of) **H**. **B** depends on the specific material you use. On the other hand, if we fix a voltage, this determines **E**, not **D**. The latter is determined by the free charge, which we do not control. **D** is therefore not particularly useful practially.

Again we must be careful about attributing the source of **H** to the free current. The curl alone doesn't determine the vector field - we also need the divergence. $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, whereas $\boldsymbol{\nabla} \cdot \mathbf{H} = -\boldsymbol{\nabla} \cdot \mathbf{M}$. Only when the divergence of **M** is zero can we make a parallel between **H** and **B**. Only when the problem has cylindrical, plane, solenoidal, or toroidal symmetry can you use Ampere's law to get **H** from the free current.

1.1.6 Linear Media

In a linear medium the magnetization is proportional to the magnetic field. This relationship is expressed as

$$\mathbf{M} = \chi_m \mathbf{H} \tag{29}$$

where the constant of proportionality is called the magnetic susceptibility. It is dimensionless, and typical values are around 10^{-5}. It is positive for paramagnets and negative for diamagnets. Note this is by convention somewhat different than the electric case, in that it it written in terms of **H** not **B**.

For linear media

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) = \mu_o(1 + \chi_m)\mathbf{H} \tag{30}$$

So

$$\mathbf{B} = \mu\mathbf{H} \tag{31}$$

where

$$\mu = \mu_o (1 + \chi_m) \tag{32}$$

μ is called the permeability of the material.

Note that at the boundary between two materials the divergence of $\mathbf{M}$ can be infinite.

The volume bound current density in a homogeneous linear medium is proportional to the *free* current density:

$$\mathbf{J}_b = \boldsymbol{\nabla} \times \mathbf{M} = \boldsymbol{\nabla} \times (\chi_m \mathbf{H}) = \chi_m \mathbf{J}_f \tag{33}$$

1.2 Ferromagnetism

Ferromagnets require no external field to sustain the magnetization - the alignment of spins is frozen into the material. The feature that distinguishes this is that neighboring dipoles interact, so that neighbors preferentially like to point in the same direction. The reasons for this are quantum mechanical. The alignment of spins is a very strong effect, and occurs in domains with sharp boundaries. The typical domain contains 10^9 spins, but the spins in neigbouring domains are in general (almost) randomly oriented with respect to one another - which is why your average nail is not magnetic.

A permanent magnet is produced by putting a ferromagnet into a magnetic field. This makes the spins at the boundaries of domains with spins aligned with the magnetic field flip, so that these domains grow and the others shrink. Eventually you end up with one domain, and the ferromagnet is saturated.

This process is not entirely reversible – when the field is switched off the ferromagnet does not become unmagnetized, but is described by a hysterisis loop. Lets assume we make the B field using some current I looped around the material, then the curve would look as follows:

The magnetization depends not only on the applied field, but also on the previous magnetic history. Customarily the loop is drawn as a plot of $\mathbf{B}$ vs $\mathbf{H}$, rather than M vs I. $\mathbf{B} = \mu_o (\mathbf{H} + \mathbf{M})$, but in practice M is huge compared to H, so that B is proportional to M.