1 Topic 3: Applications of Lagrangian Mechanics

Reading Assignment: Hand & Finch Chap. 1 & Chap. 2

1.1 Some comments on Interpretation

Conceptually, there is a fundamental difference between Newton's laws and Hamilton's principle of least action.

Newton – a local description

Hamilton–motion depends on minimizing a function of the *whole path.*

If we have a minimum principle, we should be able to get a local description:

→ consider small paths. dS must be minimized over a path small enough that only a first-order change in the potential matters. *ie.* $\vec{\nabla}V = \vec{F}$ → get Newtonian approach.

How does the particle "find the right path"?

in Newtonian approach it is easy to understand

least action - does particle test neighbouring paths to see if the have less action?

Let's look at another minimum principle – Fermat's principle:

light travels from 1 → 2 in a way that minimizes time. The amplitude at 2 is a sum of all paths – paths with different travel times have different phases, and only those near the region where phase changes slowly with path sum to a reasonable arrival amplitude.

we can see the effect when we block possible paths of light: its called diffraction.

Quantum mechanically (Feynman first realized QM could be formulated this way)-

probability a particle starts at $p1, t1$ and ends at $p2, t2$ is square of a probability amplitude. The total amplitude is the sum of amplitudes for each possible path

$$\Psi \tilde{} e^{iS/\hbar} \leftarrow \text{(fundamental action - units energy*time)}$$

in classical mechanics, $S >> \hbar$

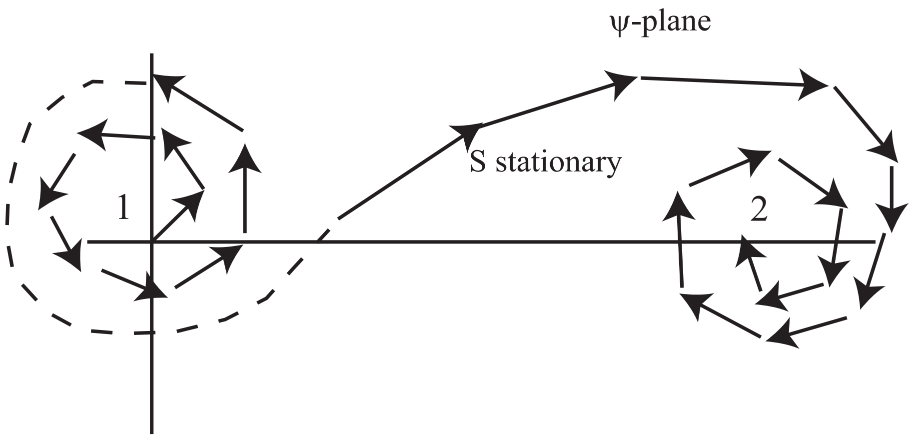
$$\Psi_2 \tilde{} \int d\tau \, e^{iS/\hbar}\Psi_1 \quad \text{(Feyman path integral)}$$

$d\tau$ is integrated over space of all possible paths. Ψ is complex – so has a phase→ for stationary paths S changes slowly wrt $\hbar$ – otherwise phases change quickly, and contributions cancel from nearby paths. The particle goes on the path for which S does not vary to first approximation.

Look at Ψ in the complex plane – only one parameter varies. Take $\Psi_1 = 1$:

We only get large contributions to the probability amplitude from paths where S is stationary (otherwise small change in path leads to large change in phase angle and they generally cancel).

This effect is stronger as S increases, because much smaller ΔS lead to large phase angles. If we block outer parts of the possible particle paths, for large S we don't effect the motion, but for small S we do → diffraction for particles.



1.2 Conserved Quantities and Symmetry Principles

Consider the case were L does not depend explicitly on one of the q_k's (where this q_k is termed 'ignorable' or cyclic). For that co-ordinate:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = 0$$

$$\frac{\partial L}{\partial \dot{q}_k} = p_k = \text{ const.of the motion}$$

in general, even if a coordinate is not ignorable, we define:

$$\frac{\partial L}{\partial \dot{q}_k} = p_k \text{ generalized or cononically conjugate momentum}$$

Lagrange's equations become:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \dot{p}_k = \frac{\partial L}{\partial q_k}$$

In the case that one of the q_k's is cyclic, we can eliminate $\dot{q}_k$ to reduce the number of variables in the problem.

An example we've already seen a few times: plane polar coordinates and central force

$$\frac{d}{dt}\left(mr^2\dot{\phi}\right) = 0 \ \rightarrow \text{angular momentum conservation}$$

this results from the symmetry of the system about the origin: since we have this symmetry L can't depend on ϕ $\rightarrow$ leads to a conservation law. Conservation laws and symmetry properties are intimately related.

1.3 Energy Conservation

Lets look at what the E-L equations say about energy. We need to know how to express the total energy in terms of L.

Consider:

$$\frac{d}{dt}(\mathbf{p}\cdot\dot{\mathbf{q}}) = \dot{\mathbf{p}}\cdot\dot{\mathbf{q}} + \mathbf{p}\cdot\ddot{\mathbf{q}} = \dot{\mathbf{q}}\cdot\frac{\partial L}{\partial \mathbf{q}} + \ddot{\mathbf{q}}\cdot\frac{\partial L}{\partial \dot{\mathbf{q}}}$$

$$\frac{d}{dt}L = \frac{\partial L}{\partial t} + \dot{\mathbf{q}}\cdot\frac{\partial L}{\partial \mathbf{q}} + \ddot{\mathbf{q}}\cdot\frac{\partial L}{\partial \dot{\mathbf{q}}}$$

$$\frac{d}{dt}(\mathbf{p}\cdot\dot{\mathbf{q}} - L) = \frac{-\partial L}{\partial t}$$

The Hamiltonian is defined as

$$H \equiv \mathbf{p}\cdot\dot{\mathbf{q}} - L$$

Treat H as a function of $\mathbf{p}, \dot{\mathbf{q}}, t$ – not $\mathbf{q}, \dot{\mathbf{q}}, t$.

If $\frac{\partial L}{\partial t} = 0$ – i.e. L is not explicitly dependent on time, then

$$\frac{d}{dt}H = 0, \quad H = const.$$

In simple cases, this is just conservation of energy. Consider cartesian coordinates and time and velocity-independent potential:

$$\mathbf{p}\cdot\dot{\mathbf{q}} = m\mathbf{v}\cdot\dot{\mathbf{r}} = m\dot{r}^2$$

$$L = \frac{1}{2}m\dot{r}^2 - V$$

$$\mathbf{p}\cdot\dot{\mathbf{q}} - L = \frac{1}{2}m\dot{r}^2 + V = const \equiv E$$

More generally, if T is a homogeneous, quadratic function of the $\dot{q}$'s

$$\dot{\mathbf{q}}\cdot\frac{\partial T}{\partial \dot{\mathbf{q}}} = 2T \quad \text{(Euler's theorem)}$$

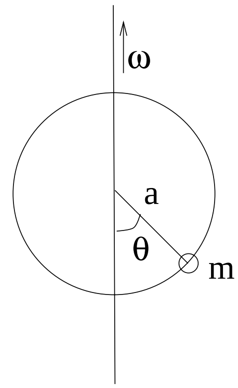
If T is a homogeneous, quadratic function of $\dot{q}$, and $V = V(\mathbf{q})$

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial T}{\partial \dot{\mathbf{q}}}$$

and

$$H = 2T - L = T + V$$

Conservation laws result from symmetries exhibited by mechanical systems - e.g. L independent of time (Hamiltonian), symmetry wrt rotation (angular momentum). This holds generally throught physics – in quantum mechanics and relativity, conservation laws are associated with symmetries in the fundamental equations.



1.3.1 Examples

1. A system with moving constraints

A massless hoop of radius a rotating about its axis, constrained to turn at constant angular velocity ω. A mass m can slide freely around the hoop. Define the angle between the vertical and a line connecting the mass to the center as θ.

$$T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\sin^2\theta\omega^2; \;\; V = -mga\cos\theta$$

The Lagrangian is

$$L = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\omega^2\right) + mga\cos\theta$$

could write Lagrange equations, but note that

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \mathbf{p}\cdot\dot{\mathbf{q}} - L = const$$

$$\begin{aligned} H &= \dot{\theta}ma^2\dot{\theta} - \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\omega^2\right) - mga\cos\theta = const \\ &= \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\sin^2\theta\omega^2 - mga\cos\theta \end{aligned}$$

Note: This is *not* the total energy, $T+V$– the middle term has the wrong sign. *Mathematically*, this is because T is not a homogeneous function in $\dot{q}^2$ (recall homogeneous to second order means $T(\lambda\dot{q}) = \lambda^n T(\dot{q})$ with $n=2$. *Physically*, there is a force doing work which produces changes in $T+V$. What is it?

We can, however, interpret L as a Lagrangian function in terms of a fixed coordinate system with the middle term regarded as an effective potential energy:

$$V_{eff}(\theta) = \frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta$$

The first term is associated with the centrifugal force which must be added to regard the rotating system as fixed. Then

$$H = T + V_{eff}$$

2. The Double Pendulum

Double pendulum with two equal masses, m, and equal lengths,l.

$$x_1 = l\sin\theta_1 \quad y_1 = l\cos\theta_1$$

$$x_2 = l(\sin\theta_1 + \sin\theta_2) \quad y_2 = l\left(\cos\theta_1 + \cos\theta_2\right)$$

$$\begin{aligned} L = &\tfrac{1}{2}ml^2[\left(\tfrac{d}{dt}\sin\theta_1\right)^2 + \left(\tfrac{d}{dt}\cos\theta_1\right)^2 + \left(\tfrac{d}{dt}(\sin\theta_1+\sin\theta_2)\right)^2 \\ &+ \left(\tfrac{d}{dt}(\cos\theta_1+\cos\theta_2)\right)^2] + mgl(2\cos\theta_1 + \cos\theta_2) \\ = &\frac{1}{2}ml^2\left[2\dot\theta_1^2 + \dot\theta_2^2 + 2\cos\left(\theta_1-\theta_2\right)\dot\theta_1\dot\theta_2\right] + mgl\left(2\cos\theta_1 + \cos\theta_2\right)\end{aligned}$$

The Lagrange equations are:

$$\theta_1: \;\; 2ml^2\ddot\theta_1 + ml^2\frac{d}{dt}\left[\cos\left(\theta_1-\theta_2\right)\dot\theta_2\right] = -ml^2\sin\left(\theta_1-\theta_2\right)\dot\theta_1\dot\theta_2 - 2mgl\sin\theta_1$$

$$\theta_2: \;\; ml^2\ddot\theta_2 + ml^2\frac{d}{dt}\left[\cos\left(\theta_1-\theta_2\right)\dot\theta_1\right] = -ml^2\sin\left(\theta_1-\theta_2\right)\dot\theta_1\dot\theta_2 - mgl\sin\theta_2$$

take the derivative......

$$\theta_1: 2l\ddot\theta_1 + l\cos\left(\theta_1-\theta_2\right)\ddot\theta_2 - l\sin\left(\theta_1-\theta_2\right)\dot\theta_2\left(\dot\theta_1-\dot\theta_2\right) = -l\sin\left(\theta_1-\theta_2\right)\dot\theta_1\dot\theta_2 - 2g\sin\theta_1$$

$$\theta_2: l\ddot\theta_2 + l\cos\left(\theta_1-\theta_2\right)\ddot\theta_1 - l\sin\left(\theta_1-\theta_2\right)\dot\theta_1\left(\dot\theta_1-\dot\theta_2\right) = l\sin\left(\theta_1-\theta_2\right)\dot\theta_1\dot\theta_2 - g\sin\theta_2$$

some algebra....

$$2l\ddot\theta_1 + l\cos\left(\theta_1-\theta_2\right)\ddot\theta_2 + l\sin\left(\theta_1-\theta_2\right)\dot\theta_2^2 + 2g\sin\theta_1 = 0$$

$$l\ddot\theta_2 + l\cos\left(\theta_1-\theta_2\right)\ddot\theta_1 + g\sin\theta_2 - l\sin\left(\theta_1-\theta_2\right)\dot\theta_1^2 = 0$$

So we get two coupled second order differential equations. We can solve these numerically for arbitrary $\theta_{1,2}$.

We can solve this exactly for small angles, and get some insight into the physics. Let $\omega_o^2 = \frac{g}{l}$, $\sin\theta\tilde{}\theta$, $\cos\theta\tilde{}1$ and ignore second order terms.

$$2\ddot\theta_1 + \ddot\theta_2 + 2\omega_o^2\theta_1 = 0$$

$$\ddot{\theta}_2 + \ddot{\theta}_1 + \omega_o^2 \theta_2 = 0$$

seek frequencies such that both masses vibrate at the same frequency, ω. Note, we are specifically seeking solutions of this type. The solution will be harmonic of the form

$$\theta_{1,2} \propto e^{i\omega t}$$

plug this into the equations above:

$$2(\omega_o^2 - \omega^2)\theta_1 - \omega^2 \theta_2 = 0$$

$$-\omega^2 \theta_1 + \left(\omega_o^2 - \omega^2\right)\theta_2 = 0$$

We can write these coupled linear equations in matrix form:

$$\begin{bmatrix} 2(\omega_o^2 - \omega^2) & -\omega^2 \\ -\omega^2 & (\omega_o^2 - \omega^2) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = 0$$

Non-trivial solutions only exist if the determinant of the coefficient vanishes (we'll see many more examples of this in the future);

$$2\left(\omega_o^2 - \omega^2\right)^2 - \omega^4 = 0$$

$$\omega^4 - 4\omega_o^2\omega^2 + 2\omega_o^4 = 0$$

$$\omega^2 = \left[2 \pm \sqrt{2}\right]\omega_o^2$$

we have two roots (ie two possible solutions for which both pendula oscillate at the same frequency);

$$\omega = [1.848, 0.765]\,\omega_0 = [\omega_+, \omega_-]$$

To get expressions for θ_1, θ_2 (the eigenvectors), substitute into the equations. We do this for both solutions.

First look at expressions for oscillation at ω_+.

$$\frac{\theta_1}{\theta_2} = \frac{\omega_+^2}{2\left(\omega_o^2 - \omega_+^2\right)} = \frac{2 - \sqrt{2}}{2\left(1 - 2 + \sqrt{2}\right)} = \frac{2 - \sqrt{2}}{2\left(\sqrt{2} - 1\right)} = \frac{1}{\sqrt{2}}$$

so

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = A_1 \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} e^{i\omega_+ t}$$

In this mode we can see that the coefficients for $\theta_{1,2}$ both have the same sign, so that the co-ordinates oscillate in the same direction. This is the "symmetric" mode.

Now look at expressions for oscillation at ω_-.

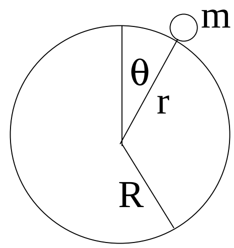
$$\frac{\theta_1}{\theta_2} = \frac{\omega_2^2}{2\left(\omega_o^2 - \omega_-^2\right)} = \frac{2 + \sqrt{2}}{2\left(1 - 2 - \sqrt{2}\right)} = \frac{-1}{\sqrt{2}}$$

$$\left[\begin{array}{c}\theta_1\\ \theta_2\end{array}\right] = A_2\left[\begin{array}{c}1\\ -\sqrt{2}\end{array}\right]e^{i\omega_- t}$$

In this mode the coefficients for $\theta_{1,2}$ have opposite signs, so the co-ordinates move in opposite directions. This is called the "antisymmetric" mode.

These are normal mode solutions – and are characteristics of linear problems. We will revisit this type of problem in much more detail later. Note: any arbitrary (small) motion can be made of the sum of motions of the two modes with appropriate amplitudes and phases.

3. A Non-holonomic constraint



Consider a point mass on a spherical bowling ball. This constraint is holonomic until the mass slides off - so in general it is non-holonomic:

$$r \geq R$$

write the Lagrangian:

$$T = \frac{1}{2}mR^2\dot{\theta}^2; \quad U = mgR\cos\theta$$

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR\cos\theta$$

The Euler-Lagrange equation is:

$$\ddot{\theta} - \frac{g}{R}\sin\theta = 0$$

Use energy conservation – T is homogeneous, quadratic function in $\dot{\theta}$. So,

$$E = T + V = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta = const$$

total energy is constant – L doesn't contain time explicitly, and the constraint is time-independent. Assume: $t = 0,\ \theta(0) = \theta_o,\ \dot{\theta}(0) = 0$. Our energy integral is only good until $\theta = \theta_m$, the point where the mass flies off the ball. Let's find out when this happens.

$$E(t = 0) = mgR\cos\theta_o$$

$$mgR\cos\theta_o = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$

$$\dot{\theta} = \sqrt{\frac{2g}{R}\left(\cos\theta_0 - \cos\theta\right)}$$

$$\tau = \sqrt{\frac{R}{2g}}\int_{\theta_o}^{\theta_m} \frac{1}{\sqrt{\cos\theta_o - \cos\theta}} d\theta$$

where τ is he time when the mass flies off. Note, we don't know what θ_m is. The beauty of the Lagrange method is that we eliminate the constraint forces from the problem. However, with what we've done so far, this leaves us no way to find them.

Invoke Newton's laws to find θ_m :

radial force = 0.

$$g\cos\theta_m = R\dot{\theta}^2|_{t=\tau}$$

$$2g(\cos\theta_o - \cos\theta_m) = g\cos\theta_m$$

$$\frac{2}{3}\cos\theta_o = \cos\theta_m$$

Finding the constraint forces motivates (in part) the method of Lagrange multipliers.

2 Lagrange Multipliers

One of the most important technical advantages of the Lagrange formulation of classical mechanics, giving wide freedom of choice in the parameters used as coordinates, is that constraints on the system, expressible as functions of the coordinates (holonomic constraints) can be automatically incorporated. We have now looked at a number of problems of this type.

There are also interesting situations where the constraints are imposed on the generalized velocities, and these cannot be integrated to give functional relationships between the coordinates. An example is a rolling constraint of a wheel on a 2D plane. The angle through which the wheel has rotated in getting from point A to point B depends not only on A and B, but on the path taken. Even imposing the constraint that the axis of the wheel be always parallel to the plane the wheel rolls on doesn't solve the problem.

Lagrange invented a method for dealing with cases where there are m constraints that can be expressed in general as

$$\mathbf{A}^{(j)} \cdot \dot{\mathbf{q}} + A_o^{(j)} = 0; \quad j = 1, ..m$$

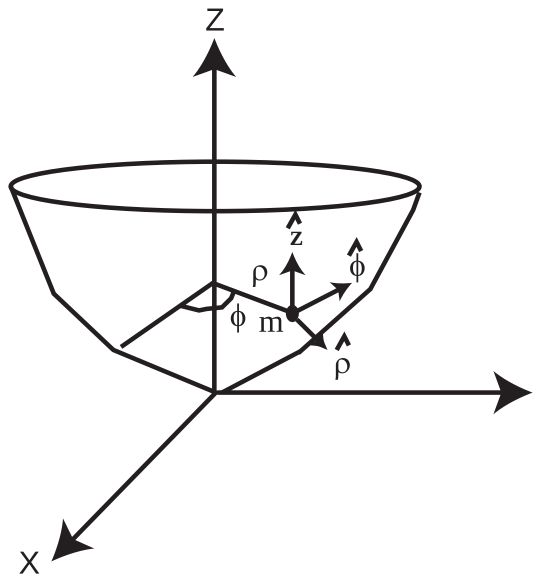
so the virtual displacements must satisfy

$$\mathbf{A}^{(j)} \cdot \boldsymbol{\delta}\mathbf{q} + A_o^{(j)} dt = 0$$

The technique of Lagrange multipliers is related to finding the forces necessary to preserve a constraint, and so it is useful even when the constraints are holonomic and we are interested in finding the constraint forces.

We will illustrate the use of Lagrange multipliers with an example where they are not required, but it will serve as an understandable example.

2.1 Lagrange Multipliers - Evil Kinievel and the Wall of Doom



m = mass of Evil and motorbike, ρ = radius vector from z–axis to Evil. The wall is a parabaloid of revolution, where the constraint that Evil stays on the wall is: $\rho^2 = az$.

$$\mathbf{v} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \frac{dz}{dt}\hat{\mathbf{z}}$$

this local coordinate system is orthonormal, so

$$T = \frac{m}{2}v^2 = \frac{1}{2}m\left[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + (\frac{dz}{dt})^2\right]$$

$$V = mgz$$

The Lagrangian is:

$$L = T - V = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + (\frac{dz}{dt})^2\right) - mgz$$

This is actually a holonomic system with two degrees of freedom. We could use the constraint equation: $\rho^2 = az$ (or in differential form $2\rho\delta\rho - a\delta z = 0$) to eliminate one variable from L. If we did this, all of our displacements (δq's) would be independent

Remember to go from

$$\int \sum_k \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right]\delta q_k\, dt = 0$$

to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

we assumed that all the δq_k's were independent. So, we must use the constraint equations (if they are holonomic) to eliminate dependant variables. This time we will use a different approach to illustrate the use of Lagrange multipliers.

2.1.1 The Method

Let's just consider a single constraint equation.

A) At a given time, we have for the virtual displacements δq (from the constraint equation)

$$\mathbf{A} \cdot \delta \mathbf{q} = 0$$

We can integrate both sides wrt time (it will be clear later why we did this)

$$\int_{t_1}^{t_2} (\mathbf{A} \cdot \delta \mathbf{q})\, dt = 0$$

B) From Hamilton's principle

$$\int_{t_1}^{t_2} dt \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} \right] \cdot \boldsymbol{\delta} \mathbf{q} = 0$$

In the past we have looked at holonomic constraints, such that we can incorporate them to reduce the number of q's to be equal to the number of degrees of freedom, giving independent δq's. This is, however, no longer the case. The δq's are related by the m constraint equations, so only $n - m$ of the δq's are truly independent.

C) Consider

$$\int_{t_1}^{t_2} \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} - \lambda \mathbf{A} \right] \cdot \boldsymbol{\delta} \mathbf{q} = 0$$

We can clearly do this, since $\mathbf{A} \cdot \boldsymbol{\delta} \mathbf{q} = \mathbf{0}$ (we've just added 0). [An aside: Here we are considering one constraint equation. We could generalize to more, and we would subtract $\sum_j \lambda^{(j)} \mathbf{A}^{(j)}$]. Suppose, as in the case of Evil, we have 3 coordinates and 1 constraint $\Rightarrow$ 2 independent coordinates. Formally we can treat q_1, q_2 as independent and q_3 as dependent.

D) *Choose* λ such that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_3} \right) - \frac{\partial L}{\partial q_3} - \lambda A_3 = 0$$

This determines λ, and we can then *solve*

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_1} \right) - \frac{\partial L}{\partial q_1} - \lambda A_1 = 0$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_2} \right) - \frac{\partial L}{\partial q_2} - \lambda A_2 = 0$$

E) We have 3 equations so far and 4 unknowns – $q_{1,2,3}, \lambda$. But, we also have the constraint equation – $\mathbf{A} \cdot \dot{\mathbf{q}} + A_0 = 0$ – a total of 4 equations and 4 unknowns.

F) Solve.......

Now, lets return to Evil.

$$L_{evil} = \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + (\frac{dz}{dt})^2 \right) - mgz$$

and

$$2\rho\delta\rho - a\delta z = 0$$

$$2\rho\dot{\rho} - a\frac{dz}{dt} = 0$$

so, we equate $A_1 = 2\rho, A_2 = 0, A_3 = -a$. Our 4 equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} - 2\lambda\rho = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} - 0 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \frac{dz}{dt}}\right) - \frac{\partial L}{\partial z} + \lambda a = 0$$

$$2\dot{\rho}\rho - a\frac{dz}{dt} = 0$$

Plugging in, these are:

$$m\left(\ddot{\rho} - \rho\dot{\phi}^2\right) = 2\lambda\rho$$

$$m\frac{d}{dt}\left(\rho^2\dot{\phi}\right) = 0$$

$$m\ddot{z} = -mg - \lambda a$$

$$2\dot{\rho}\rho - a\frac{dz}{dt} = 0$$

Now we can solve these (in principle) for $\rho(t), z(t), \phi(t), \lambda$. We know how to interpret the first three, but does λ tell us anything interesting about the physics? The physical significance is that it is related to a generalized force that maintains the constraint. A bit of geometry can be used to work out the constraint forces.

Let's look at this for an elementary case where Evil goes around in a circle of constant height: $z = const, r = const$.

$$\begin{aligned} \dot{\phi} &= \omega_o = const; \\ \dot{\rho} &= 0; \quad \rho = \rho_o \\ z_o &= \frac{\rho_o^2}{a} \end{aligned}$$

so,

$$-m\rho_o\omega_o^2 = 2\lambda\rho_o$$

$$m\rho_o^2\omega_o = l$$

$$\lambda = \frac{-mg}{a}$$

and

$$\lambda = -\frac{mg}{a} = -\frac{m}{2}\omega_o^2$$

$$\omega_o^2 = \frac{2g}{a}$$

Now lets look at how λ is related to the constraint force, $F_\perp$:

$$\tan\Psi = \frac{dz}{d\rho} = \frac{2\rho}{a}$$

$$F_\rho = -F_\perp \sin\Psi = -F_\perp \frac{2\rho_o}{\sqrt{a^2+4\rho_o^2}} = -m\rho_o\omega_o^2$$

$$F_z = F_\perp \cos\Psi = \frac{a}{\sqrt{a^2+4\rho_o^2}} = mg$$

clearly then

$$\lambda = \frac{-mg}{a} = \frac{-F_\perp}{\sqrt{a^2+4\rho_o^2}} = -\frac{1}{2}m\omega_o^2$$

which shows a connection between λ and the force of constraint.

Now lets look at at how stable Evil is. This is a nice opportunity to investigate stability to small perturbations. Suppose Evil hits a small bump which perturbs the steady motion. Does the resulting perturbation stay small, or does it grow uncontrollably? Let's assume that the bump doesn't perturb the angular momentum, so that it is the same before the bump and afterwards. So we let

$$\begin{aligned}
\rho &= \rho_0 + \delta\rho = \rho_o\left(1+\frac{\delta\rho}{\rho_o}\right)\\
z &= z_o + \delta z = z_o\left(1+\frac{\delta z}{z_o}\right)\\
\dot\phi &= \omega_o + \delta\dot\phi = \omega_o\left(1+\frac{\delta\dot\phi}{\omega_o}\right)\\
\lambda &= \lambda_o + \delta\lambda = \lambda_o\left(1+\frac{\delta\lambda}{\lambda_o}\right)
\end{aligned}$$

but,

$$l = m\rho^2\dot\phi = m\rho_o^2\omega_o$$

and

$$\rho^2 = az \text{ and } \rho_o^2 = az_o^2$$

Keep only first-order terms in small quantities and get an equation in just one of the variables, say $\delta\rho$.

$$m\left(\rho_o + \delta\rho\right)^2\left(\omega_o + \delta\dot{\phi}\right) = m\rho_o^2\omega_o \Longrightarrow \left(1 + \frac{\delta\rho}{\rho_o}\right)^2\left(1 + \frac{\delta\dot{\phi}}{\omega_o}\right) = 1$$

or

$$2\frac{\delta\rho}{\rho_o} + \frac{\delta\dot{\phi}}{\omega_o} = 0$$

and also

$$\rho^2 = az \Longrightarrow \left(\rho_o + \delta\rho\right)^2 = a\left(z_o + \delta z\right) \quad \text{or} \quad \left(1 + \frac{\delta\rho}{\rho_o}\right)^2 = \left(1 + \frac{\delta z}{z_o}\right)$$

so

$$\frac{2\delta\rho}{\rho_o} = \frac{\delta z}{z_o} = -\frac{\delta\dot{\phi}}{\omega_o}$$

$$\rightarrow \frac{2\delta\ddot{\rho}}{\rho_o} = \frac{\delta\ddot{z}}{z_o}$$

The equations of motion for ρ and z then give

$$m\delta\ddot{\rho} - m\left(\rho_o + \delta\rho\right)\left(\omega_o + \delta\dot{\phi}\right)^2 = 2\left(\rho_o + \delta\rho\right)\left(\lambda_o + \delta\lambda\right)$$

$$m\delta\ddot{z} = -a(\delta\lambda)$$

and

$$\delta\lambda/\lambda_o = -\frac{m}{a}\left(\frac{-a}{mg}\right)\delta\ddot{z} = \frac{\delta\ddot{z}}{g}$$

so

$$m\delta\ddot{\rho} - m\rho_o\omega_o^2\left(1 + \frac{\delta\rho}{\rho_o}\right)\left(1 + \frac{\delta\dot{\phi}}{\omega_o}\right)^2 = 2\rho_o\lambda_o\left(1 + \frac{\delta\rho}{\rho_o}\right)\left(1 + \frac{\delta\lambda}{\lambda_o}\right)$$

We want to get rid of $\delta\dot{\phi}$ and $\delta\lambda$ in favor of ρ:

$$m\delta\ddot{\rho} - m\rho_o\omega_o^2\left(1 + \frac{\delta\rho}{\rho_o}\right)\left(1 - \frac{2\delta\rho}{\rho_o}\right)^2 = 2\rho_o\lambda_o\left(1 + \frac{\delta\rho}{\rho_o}\right)\left(1 + \frac{\delta\ddot{z}}{g}\right)$$

Now we want to get rid of $\delta\ddot{z}$. We can use the following to do so:

$$\frac{\delta\ddot{z}}{g} = \frac{1}{g}z_o\frac{2}{\rho_o}\delta\ddot{\rho} = \frac{1}{g}\frac{\rho_o}{a}2\delta\ddot{\rho}$$

which allows us to reduce it to an equation for $\delta\rho$:

$$\delta\ddot{\rho}\left[m - 2\rho_o\lambda_o\frac{1}{g}\frac{2\rho_o}{a}\right] = -\delta\rho\left[-2\lambda_o - m\omega_o^2 + 2\left(2m\omega_o^2\right)\right]$$

$$\delta\ddot{\rho}\left[1 + \omega_o^2\frac{2\rho_o^2}{ag}\right] = -4\omega_o^2\delta\rho$$

The above is the equation for simple harmonic motion, with angular frequency ω'

$$\delta\ddot{\rho} = -\omega'^2\delta\rho$$

$$\omega' = \frac{2\omega_o}{\sqrt{1 + \frac{2\omega_o^2\rho_o^2}{ag}}}$$

which we can simplify if we use $\omega_o^2 = \frac{2g}{a}$ and $\rho_o^2 = az_o$ to:

$$\omega' = \frac{2\omega_o}{\sqrt{1 + \frac{4z_o}{a}}}$$

So, lucky for Evil, he is stable to small perturbations!