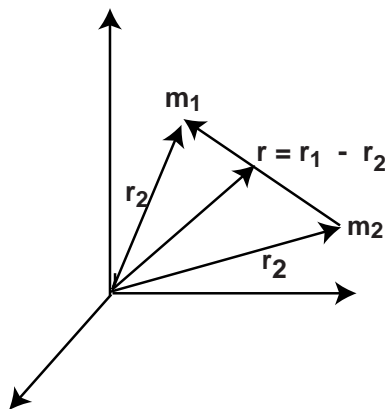


# 1 Topic 4: Two-Body Central Force Motion

Reading Assignment: Hand & Finch Chap. 4

This will be the last topic covered on the midterm exam. I will pass out homework this week but not next week.

## 1.1 Eliminating the center of mass and the equivalent 1-body problem



We will be considering the motion of two particles acting under the influence of a central force. We can eliminate the motion of the center of mass, and reduce this to an equivalent 1-body problem.

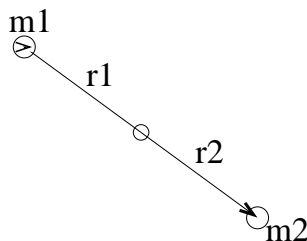
Consider the central force:  $\mathbf{F}_{12} = -\mathbf{F}_{21}$ ,  $\dot{\mathbf{p}}_1 = -\dot{\mathbf{p}}_2$ , where  $\mathbf{F} \propto \mathbf{r}_1 - \mathbf{r}_2$ . The center of mass coordinate is:

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M}; \quad M = m_1 + m_2$$

$$\frac{d^2 \mathbf{R}}{dt^2} = \frac{\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2}{M} = 0 \quad \text{and} \quad \frac{dR}{dt} = \text{const} \quad (\text{no external forces})$$

$$\mathbf{L} = \mathbf{R} \times \mathbf{P} = \text{const} \quad \text{where} \quad (\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2) \quad (\text{no external torques})$$

$\Rightarrow$  The motion of the center of mass is uniform and ignorable.



So, we will now measure  $r_1$  and  $r_2$  relative to the center of mass:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

$$T = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2$$

and  $\dot{\mathbf{R}} = 0$  in this reference frame. Conservation of linear momentum becomes

$$m_1\mathbf{r}_1 = -m_2\mathbf{r}_2$$

Since  $r_1$  and  $r_2$  are related, we have a single degree of freedom. Let's reduce down to a single position coordinate, and write an expression for the kinetic energy:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 = \left(1 + \frac{m_1}{m_2}\right) \mathbf{r}_1$$

$$\mathbf{r}_1 = \left(\frac{m_2}{m_1 + m_2}\right) \mathbf{r}$$

and

$$\mathbf{r}_2 = \left(\frac{m_2}{m_1 + m_2} - 1\right) \mathbf{r} = \frac{-m_1}{m_1 + m_2} \mathbf{r}$$

so the kinetic energy is

$$T = \frac{1}{2}m_1 \left(\frac{m_2}{m_1 + m_2}\right)^2 \dot{r}^2 + \frac{1}{2}m_2 \left(\frac{-m_1}{m_1 + m_2}\right)^2 \dot{r}^2$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{r}^2 = \frac{1}{2} \mu \dot{r}^2$$

where we define the reduced mass,  $\mu$  as  $\mu = \frac{m_1 m_2}{m_1 + m_2}$ .

We have reduced the problem to finding the motion of a single particle of mass  $\mu$  in a central field described by  $U(r)$ , so to an equivalent 1 dof problem.

## 1.2 Some Characteristics of Central Force Motion

We can derive some interesting characteristics of central force motion (for any  $U(r)$  - not just inverse square) by considering conservation of angular momentum:

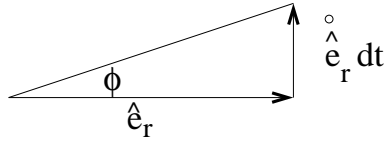
$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$

$$\dot{\mathbf{l}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}} = \mathbf{0} + \mathbf{r} \times \mathbf{F} = \mathbf{0}$$

since for a central force  $F \parallel r$ . So  $\mathbf{l} = \text{const}$  as we have previously derived.

Let  $\mathbf{r} = r\hat{\mathbf{e}}_r$  where  $\hat{\mathbf{e}}_r$  is a unit vector in the r-direction. Then,

$$\mathbf{l} = \mu r \hat{\mathbf{e}}_r \times \dot{\mathbf{r}} = \mu r \hat{\mathbf{e}}_r \times (\dot{r}\hat{\mathbf{e}}_r + r$$



$$d\phi = \frac{\left| \frac{d\hat{\mathbf{e}}_r}{dt} \right| dt}{|\hat{\mathbf{e}}_r|}$$

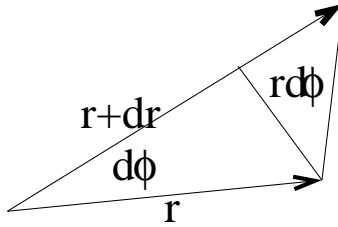
so

$$\left| \frac{d\hat{\mathbf{e}}_r}{dt} \right| = \frac{d\phi}{dt} = \dot{\phi}$$

and we have a relationship between  $\hat{\mathbf{e}}_r$ ,  $\frac{d\hat{\mathbf{e}}_r}{dt}$ ,  $\mathbf{l}$

$$|\mathbf{l}| = mr^2 \dot{\phi} = \text{const}$$

The rate of sweeping out area:



$$dA = \left( \frac{1}{2} r \right) (rd\phi) + \frac{1}{2} (rd\phi) dr$$

$$\lim_{dt \rightarrow 0} \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\phi} = \text{const}$$

So the areal velocity (rate of sweeping out area) is constant. This is Kepler's second law, which he derived empirically by observing the motion of the planets. Note though, that this is *valid for any central force*, not just  $\frac{1}{r^2}$ .

### 1.3 The Lagrangian Approach to the General Central Force Problem

Consider a conservative force:

$$\mathbf{F} = \frac{-\partial V}{\partial \mathbf{r}}; \quad V = V(r)$$

$$L = \frac{1}{2} \mu \left( \dot{r}^2 + r^2 \dot{\phi}^2 \right) - V(r)$$

The Euler-Lagrange equations are first the familiar conservation of  $l$ :

$$l = \mu r^2 \dot{\phi} = \text{const}$$

and

$$\mu \ddot{r} - \frac{l^2}{\mu r^3} = -\frac{\partial V}{\partial r}$$

We also have energy conservation, since  $\frac{\partial L}{\partial t} = 0$ , and  $T$  is homogeneous and quadratic in  $\dot{r}$

$$H = E = T + V = \text{const}$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r) = \text{const}$$

A useful way to look at this is as a "1-D" equation for motion of a particle of mass  $\mu$  in an effective potential, since the equation of motion involves only  $r$  and its derivatives;

$$E = T + V_{eff}$$

$$V_{eff} = \frac{l^2}{2\mu r^2} + V(r)$$

Here  $\frac{l^2}{2\mu r^2}$  is a "centrifugal" term arising from the kinetic energy - ie. a fictitious force due to the motion in a non-inertial frame.

We can deduce quite a bit about the character of the motion by examining the form of the effective potential. Take for example the inverse square force,

$$V = \frac{-k}{r}; \quad V_{eff} = \frac{l^2}{2\mu r^2} - \frac{k}{r}$$

Sketch the effective potential as a function of  $r$ :

$E_1 > 0$  - Unbound motion

$E_2 > 0$  - Bound orbit with turning points  $r_{\min}, r_{\max}$ , the apsidal distances

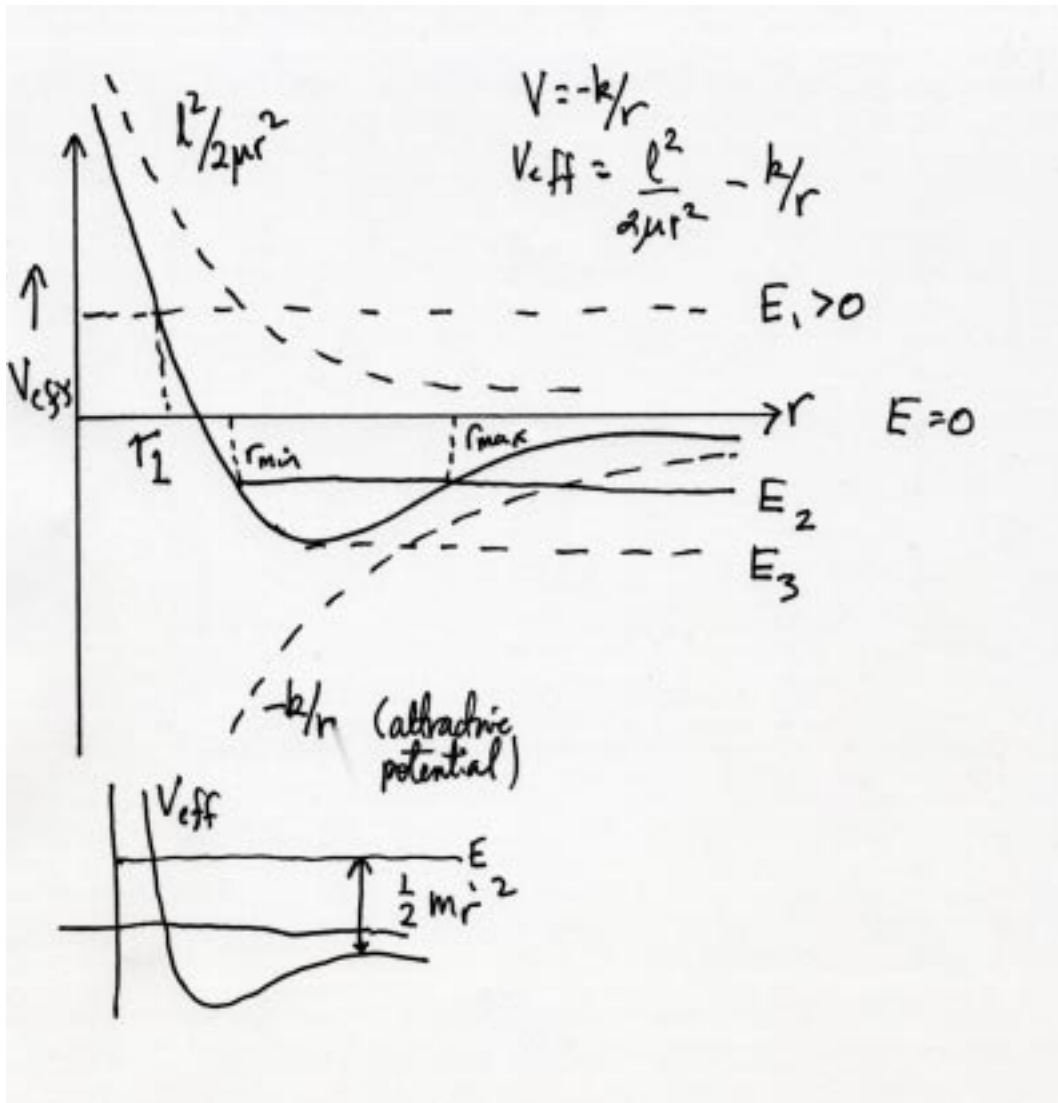
$E_3 - r$  is constant, corresponding to circular motion.  $\dot{r} = 0$ .  $E = V_{eff} = \text{const}$

$$f_{eff} = \frac{-\partial V}{\partial r} + \frac{l^2}{\mu r^3} = 0 \text{ and } \frac{\partial V}{\partial r} = \frac{l^2}{\mu r^3} \rightarrow \text{balance of centrifugal force}$$

This type of analysis is a common approach that can be applied to other-shaped potentials. Lets look at a couple of examples:

Now we will describe the orbit quantitatively. We could (as one usually does) solve the equations of motion to get  $r(t), \phi(t)$ . This constitutes a complete solution. Instead, we can make a simple transformation,  $u = \frac{1}{r}$ . We can eliminate time, and get  $r(\phi)$  - the equation for the orbit. This is more elegant.

$$\dot{r} = \dot{\phi} \frac{dr}{d\phi} = \frac{l}{\mu r^2} \frac{dr}{d\phi}$$



let  $u = \frac{1}{r}$ ,  $du = \frac{-1}{r^2} dr$

$$\dot{r} = \frac{-l}{\mu} \frac{du}{d\phi}$$

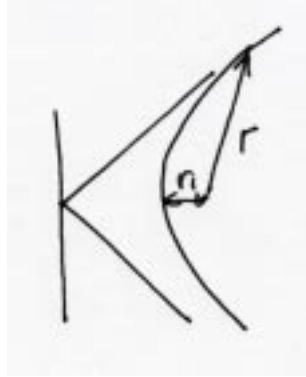
$$\ddot{r} = \frac{d}{dt} \left( \dot{\phi} \frac{dr}{d\phi} \right) = \frac{d\phi}{dt} \frac{d}{d\phi} \left( \frac{-l}{\mu} \frac{du}{d\phi} \right)$$

$$= \frac{-l^2}{\mu^2} u^2 u'' \quad \left( '' \equiv \frac{d^2}{d\phi^2} \right)$$

from the equation of motion for r:

$$\mu \ddot{r} - \frac{l^2}{\mu r^3} = -\frac{\partial V}{\partial r}$$

$$\frac{-l^2}{\mu} u^2 u'' - \frac{l^2}{\mu} u^3 = u^2 \frac{\partial V}{\partial u}$$



which we can simplify to the following diff. eq:

$$u'' + u = \frac{-\mu}{l^2} \frac{\partial V}{\partial u}$$

We have derived a differential equation for the orbit if the force law is given. For  $V \propto \frac{1}{r}$  or  $\frac{1}{r^3}$  we get a linear equation.

Note that this equation has only  $\frac{d^2}{d\phi^2}$  in it – ie. is invariant to  $\phi \rightarrow -\phi$ , and hence implies that the orbit is symmetric about the turning points (choose the turning point at  $\phi = 0$ ), and therefore the solution is independent of reflection about the apsidal vectors.

We will use this differential equation later. Now we will look at the inverse square law, but use a different approach to get  $r(\phi)$ .

## 1.4 The Inverse Square Force, $\mathbf{V} = -k/\mathbf{r}$

We will use a clever trick to get the equation for the orbit. Consider

$$\frac{d}{dt}(\mathbf{l} \times \mathbf{p}) = \left(\frac{d\mathbf{l}}{dt} \times \mathbf{p}\right) + (\mathbf{l} \times \dot{\mathbf{p}}) = \mathbf{l} \times \mathbf{F}$$

let  $\mathbf{F} = f(r)\hat{\mathbf{e}}_r$  where  $f(r) = -k/r^2$

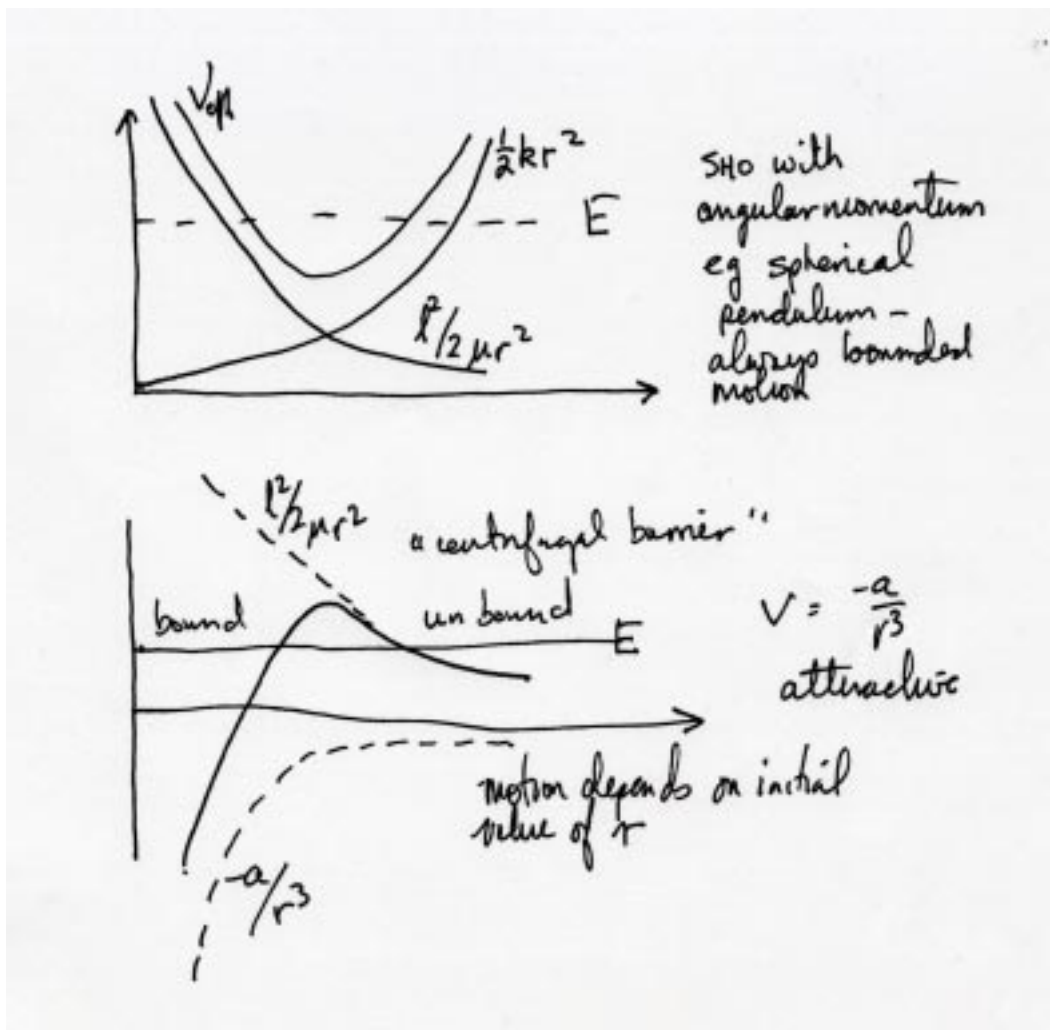
$$\mathbf{l} = \mathbf{r} \times \mathbf{p} = \mu r (\hat{\mathbf{e}}_r \times \frac{d\hat{\mathbf{e}}_r}{dt})$$

so

$$\begin{aligned} \frac{d}{dt}(\mathbf{l} \times \mathbf{p}) &= \mathbf{l} \times \mathbf{F} = \mu r^2 f(r) (\hat{\mathbf{e}}_r \times \frac{d\hat{\mathbf{e}}_r}{dt}) \times \hat{\mathbf{e}}_r \\ &= \mu r^2 f(r) \frac{d\hat{\mathbf{e}}_r}{dt} \quad \text{since } \hat{\mathbf{e}}_r \perp d\hat{\mathbf{e}}_r/dt \end{aligned}$$

Now define a vector  $\mathbf{A}$  (called the Runge-Lenz vector)

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{l} - \mu k \hat{\mathbf{e}}_r \quad (\text{Runge-Lenz vector})$$



this is a constant vector for a  $\frac{1}{r^2}$  force, since

$$\frac{d}{dt} (\mathbf{l} \times \mathbf{p} - \mu r^2 f(r) \hat{\mathbf{e}}_r) = 0 \text{ if } f(r) = -k/r^2$$

This vector lies in the plane of the orbit, since  $\mathbf{A} \cdot \mathbf{l} = 0$  - so it is a fixed vector in the orbital plane.

Take a look at the length of  $\mathbf{A}$ :

$$\begin{aligned} A^2 &= \mu^2 k^2 - \frac{2\mu k}{r} (\mathbf{p} \times \mathbf{l}) \cdot \mathbf{r} + (\mathbf{p} \times \mathbf{l}) \cdot (\mathbf{p} \times \mathbf{l}) \\ \frac{2\mu k}{r} (\mathbf{p} \times \mathbf{l}) \cdot \mathbf{r} &= \frac{-2\mu k}{r} \mathbf{l} \cdot (\mathbf{p} \times \mathbf{r}) = \frac{2\mu k}{r} l^2 = -2\mu l^2 V(r) \\ (\mathbf{p} \times \mathbf{l}) \cdot (\mathbf{p} \times \mathbf{l}) &= \mathbf{l} \cdot [\mathbf{p} \times (\mathbf{l} \times \mathbf{p})] \\ &= \mathbf{l} \cdot [p^2 \mathbf{l} - (\mathbf{p} \cdot \mathbf{l}) \mathbf{p}] \\ &= l^2 p^2 = 2\mu l^2 \frac{p^2}{2\mu} = 2\mu l^2 T \end{aligned}$$

so,

$$\begin{aligned} A^2 &= p^2 l^2 + \mu^2 k^2 - \frac{2\mu k}{r} l^2 \\ &= 2\mu l^2 (T + V) + \mu^2 k^2 \end{aligned}$$

and

$$E = \frac{A^2}{2\mu l^2} - \frac{\mu k^2}{2l^2}$$

and the constancy of  $\mathbf{A}$  implies (or is related to) energy conservation.

Now  $\mathbf{A}$  has three components, but only one can be independent, because that it is both fixed in the orbital plane, and the energy identity determines its magnitude. We can get the equation for the orbit if we consider

$$\begin{aligned} \mathbf{r} \cdot \mathbf{A} &= \mathbf{r} \cdot (\mathbf{p} \times \mathbf{l}) - \mu k \mathbf{r} \cdot \hat{\mathbf{e}}_r \\ &= l^2 - \mu k r = A r \cos \phi \end{aligned}$$

( $\phi$  is the angle between  $\mathbf{r}$  and  $\mathbf{A}$ ).

This is the polar equation for a conic section. This is a general proof of epler's first law with the eccentricity  $\varepsilon = A/\mu k$  [Kepler's first law states that the planets move in elliptical orbits with the Sun at one focus].

#### 1.4.1 Bound Orbits ( $E < 0$ )

Elliptical motion applicable to planets, satellite, etc.

For central force motion we derived

$$\frac{l^2}{\mu k r} = 1 + \frac{A}{\mu k} \cos \phi$$

where  $k = Gm_1 m_2$

The equation for an ellipse is

$$\frac{a(1 - e^2)}{r} = \frac{\alpha}{r} = 1 + \varepsilon \cos \phi$$

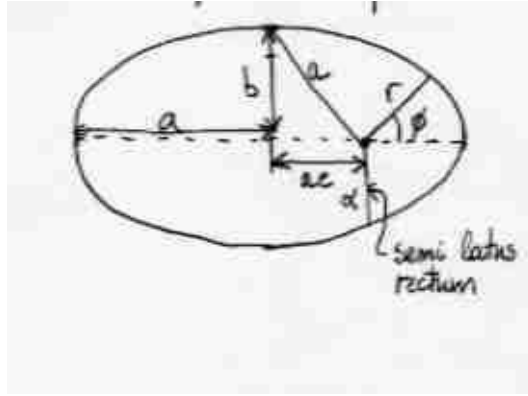
$a$  = semi major axis,  $b$  = semi minor axis, and  $b = a(1 - \varepsilon^2)^{1/2}$ . Here  $r_{\min}$  and  $r_{\max}$  are measured from the focus to the orbit.  $r_{\min, \max}$  are related to the axes of the ellipse by

$$r_{\min} = a(1 - \varepsilon) = \frac{\alpha}{1 + \varepsilon}$$

$$r_{\max} = a(1 + \varepsilon) = \frac{\alpha}{1 - \varepsilon}$$

So we can relate the parameters of the ellipse to the dynamical constants:

$$\begin{aligned} \varepsilon &= \frac{A}{\mu k} \\ \alpha &= \frac{l^2}{\mu k} \end{aligned}$$



We can also find some useful expressions for  $l, E$ :

$$\frac{l^2}{\mu^2} = a(1 - \varepsilon^2) \frac{Gm_1m_2}{\mu}$$

$$\frac{l}{\mu} = [GMa(1 - \varepsilon^2)]^{1/2} \quad (\text{specific angular momentum})$$

where  $M = m_1 + m_2$ .

Also,

$$\begin{aligned} E &= \frac{A^2}{2\mu l^2} - \frac{\mu k^2}{2l^2} \\ &= \frac{\mu k^2 \varepsilon^2 - \mu k^2}{2l^2} \\ &= \frac{-\mu k^2(1 - \varepsilon^2)}{2l^2} \\ &= \frac{-\mu k^2(1 - \varepsilon^2)}{2\mu k a(1 - \varepsilon^2)} = \frac{-k}{2a} = \frac{-Gm_2m_1}{2a} \end{aligned}$$

so,

$$\frac{E}{\mu} = -\frac{GM}{2a} \quad (\text{specific energy})$$

Finally,

$$\frac{l}{\mu} = r^2 \dot{\phi} = 2 \frac{d\mathcal{A}}{dt} = \frac{2\pi ab}{P}$$

here  $d\mathcal{A}$  is the area swept out  $-r(rd\phi)$ , and  $P$  is the orbital period. Since  $b = a(1 - \varepsilon^2)^{1/2}$ ,

$$2 \frac{d\mathcal{A}}{dt} = na^2(1 - \varepsilon^2)^{1/2}$$

where we define  $n = \frac{2\pi}{P}$ . So,

$$2 \frac{d\mathcal{A}}{dt} = (GMa)^{1/2} (1 - \varepsilon^2)^{1/2}$$

and

$$GMa = n^2 a^4; \quad GM = n^2 a^3$$

and we have shown that

$$P \propto a^{3/2} \rightarrow \text{Kepler's third law}$$

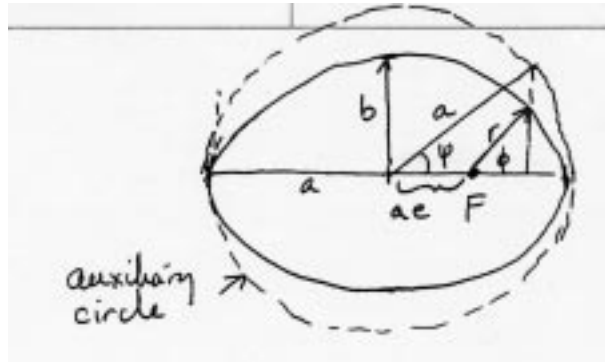
The square of the period is proportional to the cube of the major axis of the elliptical orbit. Note this is for constant  $m_1 + m_2$ , so Kepler was correct in the limit that  $m_{planet} + m_{sun} \approx m_{sun}$ . Also note that this is independent of the eccentricity of the orbit. For the Earth orbiting the Sun,  $P = 1yr$ ,  $a = 1.5 \times 10^8 km \implies M = 2 \times 10^{30} kg$ .

N.B. – The solution for inverse square force is a closed orbit. This is not true in general. The only other case is  $V \propto r^2$ , the SHO (x and y motions have the same period).

### 1.4.2 Time Dependence

We have not specified  $r(t)$ , but only derived the equation for the orbit and period. To do this we will introduce a simple geometric construct, the eccentric anomaly,  $\Psi$  (the true anomaly is generally denoted by  $\Phi$ ).

Circumscribe the ellipse of the orbit with a circle of radius  $a$  and project onto this circle:



$$a \cos \Psi = r \cos \phi + a\varepsilon$$

since we know from the equation for an ellipse that

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \phi},$$

$$\begin{aligned} a \cos \Psi &= \frac{a(1 - \varepsilon^2) \cos \phi}{1 + \varepsilon \cos \phi} + a\varepsilon \\ &= \frac{a \cos \phi + a\varepsilon}{1 + \varepsilon \cos \phi} \end{aligned}$$

$$\cos \Psi = \frac{\cos \phi + \varepsilon}{1 + \varepsilon \cos \phi}$$

and the inverse transformation:

$$\cos \phi = \frac{\cos \Psi - \varepsilon}{1 - \varepsilon \cos \Psi}$$

also we can derive the relations

$$\sin \Psi = \frac{(1 - \varepsilon^2)^{1/2} \sin \phi}{1 + \varepsilon \cos \phi}$$

$$\sin \phi = \frac{(1 - \varepsilon^2)^{1/2} \sin \Psi}{1 - \varepsilon \cos \Psi}$$

combine these:

$$\sin \Psi \sin \phi = \frac{(1 - \varepsilon^2) \sin \phi \sin \Psi}{(1 + \varepsilon \cos \phi)(1 - \varepsilon \cos \Psi)}$$

so

$$(1 + \varepsilon \cos \phi)(1 - \varepsilon \cos \Psi) = (1 - \varepsilon^2)$$

$$r = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos \phi} = a(1 - \varepsilon \cos \Psi)$$

Take the time derivative of  $\sin \phi$

$$\cos \phi \dot{\phi} = (1 - \varepsilon^2)^{1/2} \left[ \frac{\cos \Psi (1 - \varepsilon \cos \Psi) - \varepsilon \sin^2 \Psi}{(1 - \varepsilon \cos \Psi)^2} \right] \dot{\Psi}$$

$$\begin{aligned} \dot{\phi} &= (1 - \varepsilon^2)^{1/2} \frac{1}{\cos \phi} \frac{\cos \Psi - \varepsilon}{(1 - \varepsilon \cos \Psi)^2} \dot{\Psi} \\ &= (1 - \varepsilon^2)^{1/2} \frac{\dot{\Psi}}{1 - \varepsilon \cos \Psi} \end{aligned}$$

and

$$\frac{r^2 \dot{\phi}}{2} = \frac{1}{2} (1 - \varepsilon^2)^{1/2} \dot{\Psi} a^2 (1 - \varepsilon \cos \Psi)$$

The area swept out by the radius vector is

$$\frac{dA}{dt} = \frac{r^2 \dot{\phi}}{2} = \frac{na^2}{2} (1 - \varepsilon^2)^{1/2}$$

so combining with the above,

$$\frac{1}{2} (1 - \varepsilon^2)^{1/2} \dot{\Psi} a^2 (1 - \varepsilon \cos \Psi) = \frac{na^2}{2} (1 - \varepsilon^2)^{1/2}$$

$$n = \dot{\Psi} (1 - \varepsilon \cos \Psi)$$

and integrating both sides,

$$\int (1 - \varepsilon \cos \Psi) d\Psi = \int n dt$$

$$n(t - t_o) = (\Psi - \varepsilon \sin \Psi)$$

This is a complete solution, since this gives us parametric solutions for  $r(t), \phi(t)$  since we know  $r(\Psi), \phi(\Psi)$ .

## 1.5 Unbound Orbits – Scattering

$E > 0$  and attractive force:  $k > 0$

$$\mathbf{F} = -\frac{k}{r^2}\hat{\mathbf{e}}_r, \quad V = -\frac{k}{r}$$

The equation for the orbit becomes:

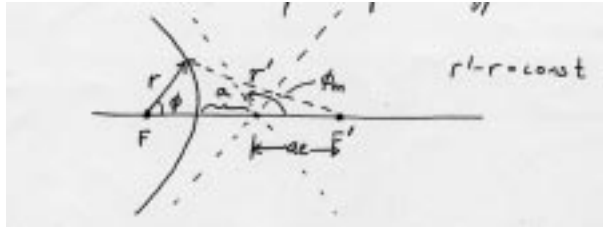
$$\frac{l^2}{\mu k} \left( \frac{1}{r} \right) = 1 + \frac{A}{\mu k} \cos \phi$$

remember  $\frac{A}{\mu k}$  is the eccentricity. We have

$$E > 0 \implies \frac{A^2}{2\mu l^2} > \frac{\mu k^2}{2l^2}$$

$$A > \mu k \implies \varepsilon > 1$$

eccentricity greater than unity describes the equation for a hyperbola:



$$r \rightarrow \infty, \quad \cos \phi_m = \frac{-\mu k}{A}$$

$$1 - \frac{\mu^2 k^2}{A^2} = \frac{2\mu l^2 E}{A^2} \implies \sin \phi_m = \sqrt{2\mu E} \frac{l}{A}$$

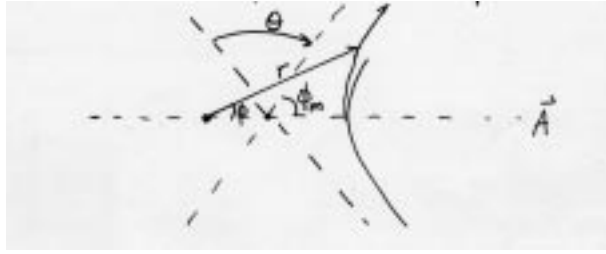
$$\cot \phi_m = -\sqrt{\frac{\mu}{2E}} \frac{k}{l}$$

which gives an equation for  $\phi_m$ .

## 1.6 Repulsive Potential – $\alpha$ particle scattering

Consider the case of a heavy nucleus, where  $m_\alpha$  is the mass of the alpha-particle (a helium nucleus = 2 protons). The nucleus is heavy (ie gold), so that  $m_\alpha \ll m_{nucleus}$ , and  $\mu \rightarrow m_\alpha$ . In this case,  $k = -2Ze^2$ , where  $Z$  is the atomic number of the heavy nucleus. The force is of course Coulomb repulsion:

$$\mathbf{F} = \frac{2Ze^2}{r^2}\hat{\mathbf{e}}_r$$



$$V = \frac{2Ze^2}{r}$$

$\mathbf{A}$  lies along the symmetry axis of the scatter. We have

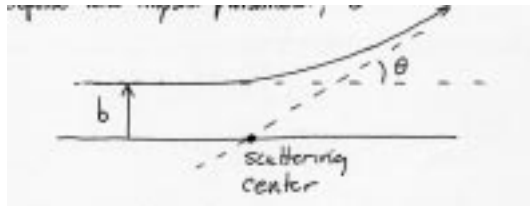
$$\frac{l^2}{\mu k} \left( \frac{1}{r} \right) = 1 + \frac{A}{\mu k} \cos \phi$$

The difference between this and the attractive case is  $k < 0$ . Define the scattering angle,  $\theta$  to be the angle between the incoming and outgoing directions.

$$\theta = \pi - 2\phi_m$$

$$\tan \left( \frac{\theta}{2} \right) = \tan \left( \frac{\pi}{2} - \phi_m \right) = \cot \phi_m = \sqrt{\frac{m_\alpha}{2E}} \frac{|k|}{l}$$

Define the impact parameter,  $b$



The angular momentum of the incoming particle with respect to the fixed scattering center is

$$l = m_\alpha v_\infty b$$

and of course

$$E = \frac{1}{2} m_\alpha v_\infty^2$$

so

$$l = \sqrt{2Em_\alpha} b$$

putting this into the expression for  $\theta$ , we get

$$\tan \left( \frac{\theta}{2} \right) = \sqrt{\frac{m_\alpha}{2E}} \frac{|k|}{b\sqrt{2Em_\alpha}} = \frac{k}{2Eb}$$

so we have a one-to-one correspondence between the impact parameter,  $b$ , and the scattering angle  $\theta$ .

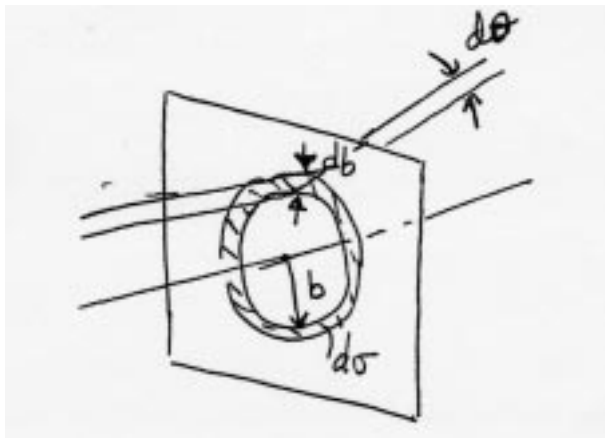
In the case of  $\alpha$ -particle scattering,  $b$  cannot be determined experimentally. In the experiment performed by Rutherford (and in general with these types of experiments), a beam of particles was incident on a thin foil, and the fraction of particles scattered through various angles,  $d\theta$  observed. This leads us to define an experimentally-motivated quantity called the *cross section*.

**Definition of cross section:**

$N = \#$  of incident particles/unit area striking a thin foil containing  $n$  scattering centers/unit area. So we deduce that

$$\frac{dN}{N} = n d\sigma$$

$d\sigma$  has the units of area,  $dN = \#$  of particles scattered through and angle between  $\theta$  and  $\theta + d\theta$ .



$$\tan \frac{\theta}{2} = \frac{k}{2Eb} = \frac{-2Ze^2}{2\left(\frac{1}{2}mv_\infty^2\right)b} = \frac{-2Ze^2}{mv_\infty^2 b}$$

take the derivative

$$\frac{1}{2} \frac{1}{\cos^2\left(\frac{\theta}{2}\right)} d\theta = \frac{2Ze^2}{mv_\infty^2 b^2} db$$

from the picture,  $d\sigma = 2\pi b db$

$$db = \frac{-mb^2 v_\infty^2}{2 \cos^2\left(\frac{\theta}{2}\right) 2Ze^2} d\theta$$

$$b = \frac{-2Ze^2 \cos \frac{\theta}{2}}{mv_\infty^2 \sin \frac{\theta}{2}}$$

so

$$\begin{aligned}
 d\sigma &= \frac{2\pi m v_\infty^2}{2 \cos^2(\frac{\theta}{2}) (2Ze^2)} \cdot \frac{(2Ze^2)^3}{(m v_\infty)^3} \cdot \frac{\cos^3(\frac{\theta}{2})}{\sin^3(\frac{\theta}{2})} d\theta \\
 &= \pi \frac{(2Ze^2)^2 \cos(\frac{\theta}{2}) \sin(\frac{\theta}{2})}{(m v_\infty)^2 \sin^4(\frac{\theta}{2})} d\theta \\
 &= \frac{1}{4} 2\pi \frac{(2Ze^2)^2 \sin \theta}{(m v_\infty)^2 \sin^4(\frac{\theta}{2})} d\theta
 \end{aligned}$$

using  $d\Omega = 2\pi \sin \theta d\theta$

$$\frac{d\sigma}{d\Omega} = \left( \frac{2Ze^2}{2m v_\infty} \right)^2 \csc^4 \left( \frac{\theta}{2} \right)$$

This is the Rutherford scattering law – Rutherford found agreement with experiment (and derived  $q_1 q_2 = 2Ze^2$ ) so long as the periaipse distance is  $\geq 10^{-12} \text{ cm}$ . Note that quantum mechanically, the concept of a cross section is still valid, but a definite trajectory is not – quantum mechanical Rutherford scattering turns out to have the same answer – fortuitously. If we try to compute the total cross section

$$\sigma_{tot} = \int 2\pi \sin \theta d\theta \frac{d\sigma}{d\Omega}$$

$\frac{d\sigma}{d\Omega} \sim \theta^{-4}$  and  $2\pi \sin \theta d\theta = d\Omega \sim \theta^2$  so it diverges at small  $\theta$  due to the infinite range of the Coulomb force. Really screening from the electrons makes this problem go away.

## 2 Astronomical Illustrations – Perturbations to the Kepler orbit

### 2.1 General Relativistic Corrections to the Planetary Orbits

One of the classic tests of general relativity is the perturbation it makes to the motion of the planets, which is measurable. GR makes a small change to the radial force. To calculate the perturbation to the orbit, return to the general orbit equation we derived earlier:

$$\begin{aligned}
 u'' + u + \frac{\mu}{l^2} \frac{dV}{du} &= 0 \\
 \frac{1}{2} u'^2 + \frac{1}{2} u^2 + \frac{\mu}{l^2} V &= \text{const} \\
 T &= \frac{1}{2} \mu \dot{r}^2 + \frac{l^2}{2r^2 \mu}
 \end{aligned}$$

from before we found that  $\dot{r} = -\frac{l}{\mu} u'$  so

$$T = \frac{1}{2} \frac{l^2}{\mu} u'^2 + \frac{1}{2} \frac{l^2}{\mu} u^2$$

so

$$\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{\mu}{l^2}V = \frac{\mu}{l^2}E$$

and

$$u' = \left[ \frac{2\mu}{l^2}(E - V) - u^2 \right]^{1/2}$$

Integrate between perihelia (note now the orbit is *not* necessarily closed, so perihelion can happen at different  $\phi$ )

$$\Delta\phi = \oint \frac{du}{\left\{ \frac{2\mu}{l^2}(E - V) - u^2 \right\}^{1/2}}$$

if  $V = \frac{-Gm_1m_2}{r}$  then  $\Delta\phi = 2\pi \implies$  we have a closed orbit for a pure  $1/r$  potential.

let  $V \rightarrow V_o + \delta V$ ,  $\Delta\phi \rightarrow 2\pi + \delta\phi$

$$\Delta\phi + \delta\phi = \oint du f|_{V_o} + \oint du \frac{\partial f}{\partial V}|_{V_o} \delta V + \text{higher order}$$

so

$$\delta\phi \approx \oint \frac{\partial}{\partial V} \left( \frac{l}{\{2\mu(E - V) - l^2u^2\}^{1/2}} \right) du$$

note, we can change  $\frac{\partial}{\partial V}$  to  $\frac{\partial}{\partial l} \rightarrow$  they differ by  $\frac{lu^2}{\mu}$  so:

$$\delta\phi \approx \frac{\partial}{\partial l} \oint \frac{\delta V \mu du}{u^2 \{2\mu(E - V_o) - l^2u^2\}^{1/2}}$$

Now we do a trick – convert this to an integral over  $\phi$  around the unperturbed orbit: (note, the denominator of the integrand is referenced to the *unperturbed* orbit)

$$\begin{aligned} \frac{du}{d\phi} &= \frac{1}{l} \{2\mu(E - V_o) - u^2\}^{1/2} \\ \delta\phi &\approx \frac{\partial}{\partial l} \oint \frac{\delta V \mu du}{u^2 l \left(\frac{du}{d\phi}\right)} = \mu \left[ \frac{\partial}{\partial l} \oint \frac{\delta V(r) r^2 d\phi}{l} \right] \end{aligned}$$

Let the perturbation be of the form  $\delta V(r) = \gamma/r^3$ . Using the equation for an elliptical orbit

$$\begin{aligned} \frac{1}{r} &= \frac{\mu k}{l^2} (1 + \varepsilon \cos \phi) \\ \delta\phi &= \gamma \mu \frac{\partial}{\partial l} \int_0^{2\pi} \frac{\mu k}{l^3} (1 + \varepsilon \cos \phi) d\phi \\ &= \gamma \mu^2 2\pi k \frac{\partial}{\partial l} (l^{-3}) = \frac{-6\pi \mu^2 k \gamma}{l^4} \end{aligned}$$

General relativity gives a perturbation of this form to Newtonian gravity, with

$$\gamma = \frac{-l}{\mu} \left( \frac{GM_{sun}}{c^2} \right)$$

so

$$\begin{aligned}\delta\phi &= 6\pi \frac{\mu^2 l^2}{\mu l^4} \left( \frac{GM_{sun}}{c^2} \right) GM\mu \\ &= 6\pi \frac{\mu^2 (GM_{sun})^2}{l^2 c^2} \text{ where } [M \approx M_{sun}]\end{aligned}$$

from before, we derived the specific angular momentum;

$$\frac{l}{\mu} = [GMa(1 - \varepsilon^2)]^{1/2}$$

Using this above we get

$$\delta\phi = \frac{6\pi GM_{sun}}{a(1 - \varepsilon)c^2} \text{ per orbit}$$

Note, the perturbation is the biggest for the largest  $\varepsilon$  and the smallest  $a$ .

For mercury, the contribution to the precession of its perihelion due to GR is  $43''/\text{century}$ . There is a  $532''/\text{century}$  precession due to other Newtonian perturbations which can be calculated. The GR term agrees with measurement to 0.4%, which is considered an important confirmation of the theory.

It was suggested by Dicke and others that other effects may contribute, making agreement with GR coincidental. One of these effects is the possibility of a non-spherical sun. This has, however, been shown not to be an important effect.