1 Topic 5: Oscillations

Reading Assignment: Hand & Finch Chap. 3

A system that is in a configuration of stable equilibrium will undergo oscilliatory motion, which can be described by simple harmonic oscillation if the displacements are small enough. One can always consider displacements small enough that non-linear terms in the equations can be ignored.

First, lets be clear about what we mean by stable equilibrium. An equilibrium point is a configuration of the system where the forces vanish:

$$\frac{\partial V}{\partial \vec{q}} = 0$$

For the equilibrium to be stable, we need the additional condition that the equilibrium point be at a potential minimum - ie the second derivative of the potential be positive;

$$\frac{\partial^2 V}{\partial q^2} > 0$$

. Take for example a pendulum, which has two equilibrium points, at $\theta = 0, \pi$ – only the $\theta = 0$ point is stable.

2 The Linear Oscillator

Consider a mechanical system with a Lagrangian, L . For now consider a single variable. We can expand L in powers of $q, \dot{q}$:

$$L = a + bq + c\dot{q} + dq^2 + eq\dot{q} + f\dot{q}^2 + ...$$

Here $a, b, c....$ are constants that we can derived (by definition of the Taylor expansion) from the partial derivatives of L : $b = \frac{\partial L}{\partial q}|_{q_o,\dot{q}_o}$. Now consider the system to be in equilibrium (either stable or unstable) such that $b = 0$ (by definition). If we keep only terms up to second order, from the E-L equations, the EOM becomes:

$$\frac{d}{dt}(c + eq + 2f\dot{q}) = 2dq + e\dot{q}$$

$$\ddot{q} - \frac{d}{f}q = 0$$

and the motion depends only on the ratio $d/f \equiv \omega_o^2$. Now $f > 0$, since it represents the factor in front of the kinetic energy term in the Lagrangian, which must be positive for a physical system. So, we find that the sign of ω_o^2 depends on the sign of $d = \frac{\partial^2 L}{\partial q^2}$. If $d < 0$, then $\omega_o^2 > 0$, and we have the equation for a free oscillator:

$$\ddot{q} + \omega_o^2 q = 0$$

which we know has solutions of the form $q(t) = Ae^{\pm i\omega_o t}$.

[A side note: we are going to use complex notation to simplify the algebra. The real, physical solution is given by taking the real part. The most general complex solution is

$$q(t) = Ae^{i\omega_o t} = A\left(\cos\left(\omega_o t + \phi\right) + i\sin\left(\omega_o t + \phi\right)\right)$$

, where $A = A_o e^{i\phi}$ is a comlex constant. The physical interpretation of the real and imaginary parts of A is the displacement and velocity at $t = 0$.

$$\begin{aligned} q(0) &= \mathrm{Re}[A] \\ \dot{q}(0) &= \mathrm{Re}[iA] = -\mathrm{Im}[A] \end{aligned}$$

so A contains the information about the initial conditions;

$$A = q(0) - i\dot{q}(0)$$

shifting the initial conditions to another time means multiplying A by a complex phase factor.

Now lets just consider a holonomic, conservative system where the potential energy depends only on q, and the kinetic energy is homogeous to degree 2 in the $\dot{q}$. In this case$\frac{\partial^2 L}{\partial q^2} = \frac{-\partial^2 V}{\partial q^2}$, and our condition for simple harmonic oscillation comes down to one of being at a stable equilibrium position. Now if we are at an unstable equilibrium point, $\frac{\partial^2 V}{\partial q^2} < 0$, and $\omega_o^2 < 0$, and our soltions are of the form $q(t) = Ae^{\omega_o t} + Be^{-\omega_o t}$. In other words, with a small perturbation, the solution grows exponentially (until our original small displacement assumption is no longer valid).

This week, we will consider additional terms in this equation that represent damping, driving forces, and time-dependent coefficients.

2.1 The Damped Simple Harmonic Oscillator

Lets rescale time so that $\omega_o \equiv 1$. We can always do this, and at the end of the problem rescale back. Lets consider a frictional force that we can express by the following

$$F_{friction} = \frac{-1}{Q}\dot{q}$$

Examples of such a force might be a pendulum immersed in liquid, electrical circuits with resistance, etc.. The EOM becomes

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = 0$$

We quess that the solution will have the form

$$q(t) = e^{i\alpha t}$$

Substitute this into the EOM:

$$\left(-\alpha^2 + i\frac{\alpha}{Q} + 1\right) e^{i\alpha t} = 0$$

$$\alpha = \frac{i}{2Q} \pm \sqrt{1 - \frac{1}{4Q^2}}$$

The roots can be either real or complex, depending on Q. Once we have a complex solution that works, the math guarantees that no other solutions will exist, since q, q^* are linearly independent: a second order ODE must have two independent solutions (we specify two initial conditions, $q(0), \dot{q}(0)$.

2.1.1 Underdamped motion: Q>$\frac{1}{2}$

If $Q > \frac{1}{2}$, then the radical is <0, and we have

$$q(t) = A_c e^{-\frac{t}{2Q}} e^{\pm i\omega' t}$$

where

$$\omega' \equiv \sqrt{1 - \frac{1}{4Q^2}}$$

and

$$\mathrm{Re}\left[q(t)\right] = Ae^{-\frac{t}{2Q}} \sin\left(\omega' t + \phi\right)$$

So we can see that the amplitude is exponentially damped. The smaller Q is, the more quickly the oscillations will die out. Also the frequency is smaller than for the undamped case.

2.1.2 Over damped: Q<$\frac{1}{2}$

In this case the radical is greater than 0, and our solutions look like:

$$q(t) = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

$$\lambda_\pm = \frac{-1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1} < 0$$

So $\lambda_\pm$ are both < 0, and the motion dies out before the solution can pass through 0. A, B are set by the initial conditions: $q(0) = A + B$ and $\dot{q}(0) = -\left(\lambda_+ A + \lambda_- B\right)$

2.1.3 Critically damped: Q=$\frac{1}{2}$

If $Q = 1/2$,

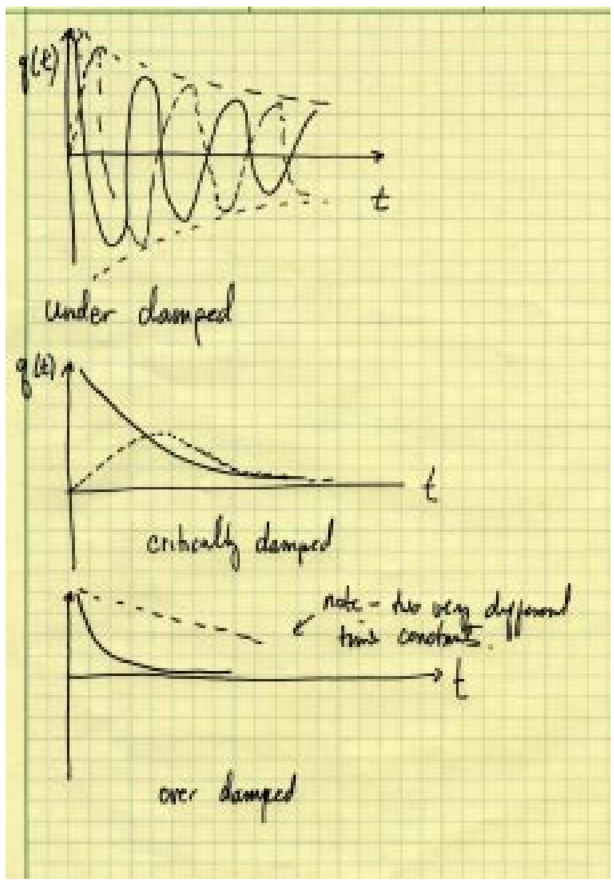
$$q(t) = e^{\lambda t}$$

where $\lambda = -1$. Since $q(0), \dot{q}(0)$ must both be specified, we need a second solution to the quadratic equation. Assume $Q = \frac{1}{2} - \varepsilon$ (very slightly overdamped). From the overdamped solution, assume $B = -A$, and let $\varepsilon \to 0$. In this limit, the solution is proportional to te^{-t}.

$$q(t) = Ce^{-t} + Dte^{-t}$$

C,D are determined by the initial conditions. If we want e.g. a sensitive seismograph $\to$ must return to 0 quickly after a tremor, we want critical damping.

Our solution look like:



Lets look at the physical meaning of Q:

In the case of the underdamped oscillator: the amplitude is proportional to $e^{-\frac{t}{2Q}}$ (neglecting the small frequency shift). The energy is proportional to the amplitude squared. We can see that the # of periods it takes to decay to $\frac{1}{e}$ of the original value is $\frac{Q}{2\pi}$ oscillations

$$\frac{\Delta E}{E} = \frac{-2\pi}{Q}$$

where $\Delta E/E$ is the fraction of energy lost/cycle.

$$\frac{dE}{dt} = -\frac{E}{Q}$$

which shows the stored energy decays exponentially.

Some physical examples of typical Q's:

50g mass hanging from a coil spring	~25
Earth, for oscillations due to quake	~200
FM radio receiver	5×10^3
tuning fork	10^4
sodium atom emitting yellow light	5×10^7
superconducting rf cavity	10^{10}

2.2 Driven Oscillations

Consider a driving force independent of the system, $F = F(t)$. If we considered $F = F(q, t)$ it would be too complicated.

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = F(t)$$

This equation is an inhomogeneous (RHS $\neq 0$) linear equation. Note the system energy is no longer constant. We are interested in how energy is stored (absorbed) by the system

We no longer can create new solutions by arbitrary linear combinations of solutions. Instead, if we know or guess one solution, the "steady state" solution, the most general solution is found by adding a solution of the free oscillator

$$\begin{aligned} q_{general} &= q_{steady\,state} + q_{transient} \\ &= particular + free \end{aligned}$$

We can solve generally using Green's functions. Now we will guess the solutions for two illustrative cases:

1) Step function with a driving force

The force changes from 0 (t<0) to F_o(t>0)

Guessing the steady state solution – the force will induce oscillations which die down due to damping. The steady state will be the system at rest at a new equilibrium.

Match boundary conditions→

$q, \dot{q}$ must be continuous, $\ddot{q}$ has a jump of F_o at $t = 0^+$. Take $F_o = 1$ (we can always scale the solution)

Dual continuity requirements give two equations in two unknowns which completely determines the solution. We get these by matching boundary conditions

transient solutions:

$$\begin{aligned} q_1 &= e^{-\frac{t}{2Q}}\cos\omega' t \quad ; \quad q_1(0) = 1, \dot{q}_1(0) = -\frac{1}{2Q} \\ q_2 &= e^{-\frac{t}{2Q}}\sin\omega' t \quad ; \quad q_2(0) = 0, \dot{q}_2(0) = \omega' \end{aligned}$$

$q(t) = q_{steady} + q_{transient} = F_o + q_{transient} = 1 + q_{transient}$ since we have $q = F_o$ at $t \to \infty$ and $\dot{q}, \ddot{q} = 0$)

match boundary conditions for the transient solutions:

$$q\left(0^+\right) = 1 + aq_1\left(0\right) + bq_2\left(0\right) = 0 \quad \text{continuity of q}$$

$$\dot{q}\left(0^+\right) = a\dot{q}_1\left(0\right) + b\dot{q}_2\left(0\right) = 0 \text{ continuity of } \dot{\text{q}}$$

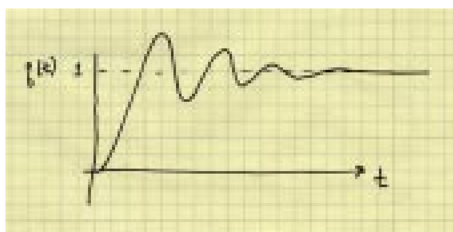
Take boundary conditions $q(0^+) = \dot{q}\left(0^+\right) = 0$

$$1 + a = 0$$

$$-\frac{1}{2Q}a + \omega' b = 0$$

so

$$q(t) = 1 - e^{-\frac{t}{2Q}}\left(\cos\omega' t - \frac{1}{2Q\omega'}\sin\omega' t\right)$$



2.3 A Sinusoidal Driving Force

Here we don't care about the transient solution, but just want to know what happens in steady state

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = Ae^{i\omega t}$$

where $Ae^{i\omega t}$ is called a forcing term. We seek steady-state (particular) solution of the form $q\tilde{\ } e^{i\omega t}$

$$q = \frac{Ae^{i\omega t}}{\left(-\omega^2 + \frac{i\omega}{Q} + 1\right)} + q_{transient}$$

where $q_{transient}$ dies out in time.

$$q = Ae^{i\omega t}\frac{\left[\left(1-\omega^2\right) - \frac{i\omega}{Q}\right]}{\left(1-\omega^2\right)^2 + \frac{1}{Q^2}\omega^2}$$

The magnitude of q is given by

$$|q| = |qq^*|^{1/2} = \frac{A}{\left[\left(1-\omega^2\right)^2 + Q^{-2}\omega^2\right]^{1/2}}$$

The denominator will be minimized when

$$2\left(1-\omega^2\right) = Q^{-2} \rightarrow resonance$$

$$\omega^2 = 1 - \frac{1}{2}Q^{-2}$$

now $\omega_o^2 = 1$; $\omega^2 = \omega_o^2 - \frac{1}{2}Q^{-2}$ so damping lowers resonance frequency.

For $Q^{-1} << \omega_o$ (ie large Q's)

$$\omega = \omega_o - \frac{1}{4}\frac{Q^{-2}}{\omega_o}$$

resonance:

$$\begin{aligned} |q| &= \frac{AQ}{\left(\omega_o^2 - \frac{Q^{-2}}{4}\right)^{1/2}} \\ &\approx \frac{AQ}{\omega_o^2} \qquad (Q^{-2} << \omega_o^2) \end{aligned}$$

Look at the FWHM (width of the resonance):

FWHM⇒denominator^2 = 2 times the value at resonance

$$\left(\omega_o - \omega_{1/2}^2\right)^2 + Q^{-2}\omega_{1/2}^2 = 2Q^{-2}\left(\omega_o^2 - \frac{1}{4}Q^{-2}\right)$$

for $\omega \approx \omega_o$, $Q^{-1} << \omega_o$

$$\begin{aligned} \left(\omega_o^2 - \omega^2\right)^2 + \frac{\omega^2}{Q^2} &= (\omega_o - \omega)^2(\omega_o + \omega)^2 + \frac{\omega^2}{Q^2} \\ &\approx 4\omega_o^2\left((\omega_o - \omega)^2 + \frac{Q^{-2}}{4}\right) \end{aligned}$$

$$\Longrightarrow 4\omega_o^2\left[\left(\omega_o - \omega_{1/2}\right)^2 + \frac{Q^{-2}}{4}\right] \simeq 2Q^{-2}\omega_o^2$$

$$\left(\omega - \omega_{1/2}\right)^2 \simeq \frac{1}{4Q^2}$$

and

$$\omega_{1/2} \simeq \omega_o\left(1 \pm \frac{1}{2Q}\right)$$

$$\frac{FWHM}{\omega_o} = \frac{2\left(\omega_{1/2} - \omega_o\right)}{\omega_o} = \frac{1}{Q}$$

The fraction of energy lost/cycle of free oscillation is:

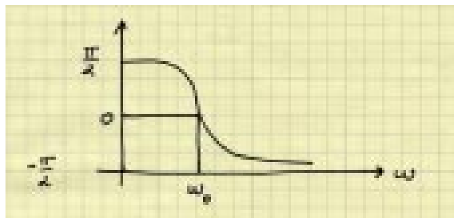
$$\frac{\Delta E}{E} = \frac{Q^{-1}\left\langle \dot{q}^2 \right\rangle \frac{2\pi}{\omega_o}}{\left\langle \dot{q}^2 \right\rangle} = \frac{2\pi}{Q}$$

The phase velocity is:

$$\tan\phi = \frac{\mathrm{Im}\,(q)}{\mathrm{Re}\,(q)} = phase\ of\ q$$

$$\text{phase of } \dot{q} = \tan\left(\phi + \frac{\pi}{2}\right) = \frac{\omega/Q}{\omega_o^2 - \omega^2} \tilde{} \frac{Q}{\omega_o - \omega}$$

and the phase switches sign passing through resonance.



3 Phase Diagrams

The state of a one-dimensional oscillator is comletely specified if we are given the co-ordinate and its time derivative at one point in time: $q(t_o), \dot{q}(t_o)$ (two quantities are needed since we have a second order equation). We can consider $q(t_o), \dot{q}(t_o)$ to be the coordinates of a point in two dimensional *phase space.* For n degrees of freedom phase space is $2n$ dimensional, however for a 1-D oscillator it is a plane. The point $P(q(t_o), \dot{q}\,(t_o))$ describing the state of the oscillating system moves around in phase space with time, describing different paths in phase space. The totality of all possible phase paths constitutes the *phase portrait*, or *phase diagram* of the oscillator.

If we look at the 1-D oscillator, the solutions (taking the real part) are:

$$q(t) = q_m \sin\left(\omega_o t + \phi\right)$$

$$\dot{q}(t) = q_m \omega_o \cos\left(\omega_o t + \phi\right)$$

and so

$$\frac{q^2}{q_m^2} + \frac{\dot{q}^2}{q_m^2 \omega_o^2} = 1$$

This describes a family of ellipses, where each ellipse corresponds to a different energy: $E = \frac{1}{2} k q_m^2$

$$\frac{q^2}{2E/k} + \frac{\dot{q}^2}{2E\omega_o^2/k} = 1$$

so each phase path corresponds to a particular total energy.

Note: no two phase paths can cross, or for a given set of initial conditions the motion could proceed along different phase paths. This is however impossible, since the solution to the differential equation is unique.

If we plot the axes as in the picture, the motion of a point $P(q, \dot{q})$ will always be clockwise, since for $q > 0$, $\dot{q}$ is always decreasing, and for $q < 0$, $\dot{q}$ is always increasing.

Note that in order to get the solutions $q(t), \dot{q}(t)$, we had to integrate a second order differential equation. We can, however, get the trajectory in phase space more simply, since

$$\ddot{q} + \omega_o^2 q = 0$$

and

$$\frac{d\dot{q}}{dq} = -\omega_o^2 \frac{q}{\dot{q}}$$

this is a first order differential equation for $\dot{q} = \dot{q}(x)$. The solution is just the equation for the ellipse we got above. While the general solution we get for this simple case by solving the second order equation is simple to arrive at, it is sometimes much easier to get the equation for the phase path with solving directly for $q(t)$.