# 1 Topic 7: Rotating Co-ordinate Systems

Reading assignment: Hand and Finch Chapter 7

Earth is not an inertial frame. If we were observers on the Sun we would see it racing along at 66,700 mph in an elliptical orbit. An observer at Earth's center would see you rotating at 1,038 mph. Yet, as far as we are concerned it seems nearly inertial, with deviations difficult to detect.

In this section we will develop a systematic way of translating back and forth between inertial and rotating frames. This is purely a mathematical process.

What we want to investigate:

1. - Given fixed rotations, how to transform vectors between frames.
   - How to transform co-ordinates of a fixed point between two rotating frames.
   - How to calculate the time derivatives of vectors in one frame and relate them to another non-inertial frame.

If we can do the above, we can calculate observables in rotating frames. Specifically, we can find the "fictitious" forces due to the rotation of the co-ordinate system.

## 1.1 What is a Vector?

A vector is "a quantity that has both direction and magnitude". If we want to refer to a physical vector without reference to a co-ordinate system, we will use $\mathbf{r}$. Often we need to work with coordinates of vectors, $\mathbf{a} : (a_1, a_2, a_3)$ where these clearly depend on reference frame. In terms of unit vectors along the $x, y, z$ axes we express $\mathbf{a}$ as:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

and

$$a^2 = a_k a_k$$

If we go to a different co-ordinate system, we express $\mathbf{a}$ as

$$\mathbf{a} = a_1^{'}\mathbf{i}' + a_2^{'}\mathbf{j}' + a_3^{'}\mathbf{k}' = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

and of course the length is preserved:

$$a_k a_k = a_k^{'} a_k^{'}$$

More generally, the scalar product of any two vectors is preserved upon co-ordinate transformation (it is after all just a number)

$$a_k b_k = a_k^{'} b_k^{'}$$

We can relate the $a_k$'s to the $a_k^{'}$'s:

$$
\begin{aligned}
a_1^{'} &= \mathbf{i}' \cdot \mathbf{a} = a_1 \mathbf{i}' \cdot \mathbf{i} + a_2 \mathbf{i}' \cdot \mathbf{j} + a_3 \mathbf{i}' \cdot \mathbf{k} \\
a_2^{'} &= \mathbf{j}' \cdot \mathbf{a} = a_1 \mathbf{j}' \cdot \mathbf{i} + a_2 \mathbf{j}' \cdot \mathbf{j} + a_3 \mathbf{j}' \cdot \mathbf{k} \\
a_3^{'} &= \mathbf{k}' \cdot \mathbf{a} = a_1 \mathbf{k}' \cdot \mathbf{i} + a_2 \mathbf{k}' \cdot \mathbf{j} + a_3 \mathbf{k}' \cdot \mathbf{k}
\end{aligned}
$$

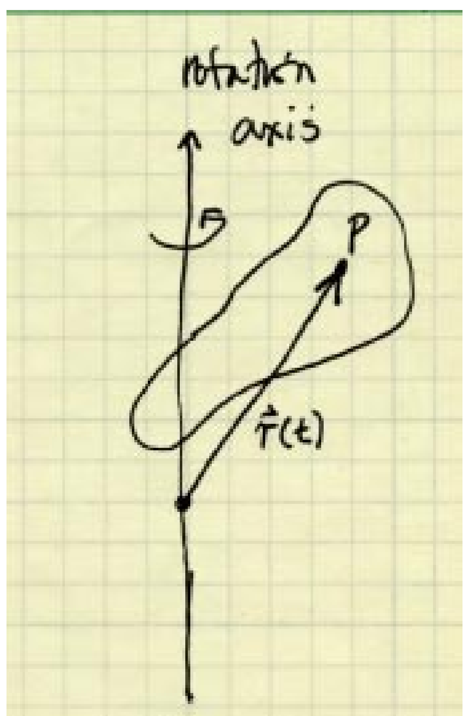
where $\mathbf{i}' \cdot \mathbf{i}$ is the projection of the $x'$ axis onto the $x$ axis, etc.. The primed co-ordinates are given in terms of the unprimed co-ordinates by fixed coefficients depending only on the relative orientation of the co-ordinate systems, as we would expect.

We can in fact *define* a vector as something that transforms according to the above. This definition is preferable since it i) distinguishes vectors from tensors, and ii) avoids the vague concepts of direction and magnitude.

## 1.2 Infinitesimal Rotations of a Rigid Body and Angular Velocity

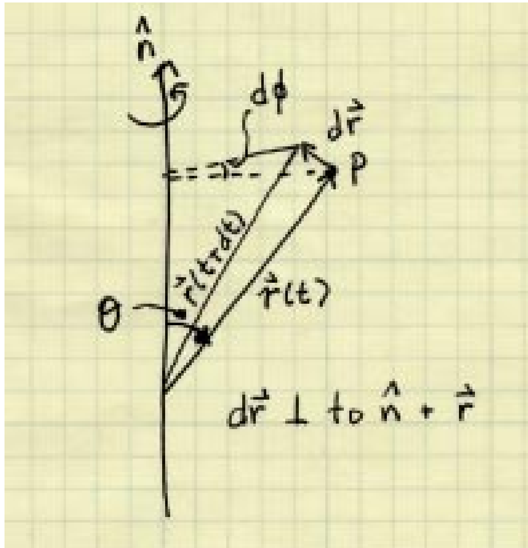
We define a rigid body to be a collection of points held together at fixed distances. A rigid body has 6 degrees of freedom; 3 to fix a single point in space, and 3 to describe the orientation.

Lets look at the rotational motion by considering the infinitesimal rotation of th body with one point fixed with the origin on the rotation axis.




The body rotates by $d\phi$ about the axis $\hat{\mathbf{n}}$ in a time $dt$

$$\mathbf{r}\,(t + dt) \equiv \mathbf{r}(t) + \mathbf{dr}$$




$$|\mathbf{dr}| = r \sin\theta \, d\phi$$

and $\mathbf{dr}$ lies perpendicular to $\hat{\mathbf{n}}$ ($\mathbf{d}\boldsymbol{\phi}$) and to $\mathbf{r}$. Now

$$|\hat{\mathbf{n}} \times \mathbf{r}| = r \sin\theta$$

$$\Rightarrow \mathbf{dr} = d\phi \, \hat{\mathbf{n}} \times \mathbf{r}$$

where $\mathbf{d}\boldsymbol{\phi} \equiv \hat{\mathbf{n}} \, d\phi$ and for infinitesimal rotations only

$$\mathbf{dr} = \mathbf{d}\boldsymbol{\phi} \times \mathbf{r}$$

The velocity of any point $P$ in the rotating body in terms of the angular velocity $\omega$ is:

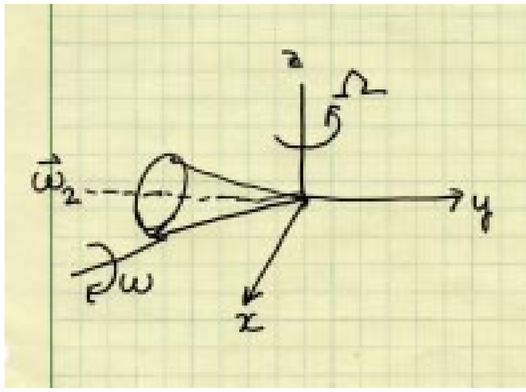
$$\frac{d\mathbf{r}}{dt} = \mathbf{v}_p = \boldsymbol{\omega} \times \mathbf{r} \qquad (\boldsymbol{\omega} \equiv \frac{d\boldsymbol{\phi}}{dt})$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity. Now $\mathbf{d}\boldsymbol{\phi}$ and $\boldsymbol{\omega}$ are not really vectors, but "pseudo vector". The axis of rotation defines a plane in which rotation takes place, and so has a handedness or helicity. This is not the same as a direction to a point located in space. These are also called axial vectors; they rotate like vectors but are invariant under reflections. Any cross product between two vectors is a pseudo vector, since if we have $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \to -\mathbf{a}$, $\mathbf{b} \to -\mathbf{b}, \mathbf{a} \times \mathbf{b}$ is unchanged.

## 1.3 Instantaneous Axis of Rotation

We can reduce two or more simultaneous rotations to a single rotation if we consider a short time interval $dt$. We can see this intuitively if we consider a cone rolling on a horizontal surface:

$$\boldsymbol{\omega} = -\,|\omega|\,(\cos(\Omega t)\,\mathbf{i} + \sin(\Omega t\,)\mathbf{j})$$




We can show that the angular velocities add like vectors:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$

$$\begin{aligned}
\mathbf{r}(d\phi_1, d\phi_2) - \mathbf{r}\,(0,0) &= d\phi_1 \frac{\partial \mathbf{r}}{\partial \phi_1} + d\phi_2 \frac{\partial \mathbf{r}}{d\phi_2} + \mathcal{O}\left(d\phi^2\right) \\
(\mathbf{dr} &= \mathbf{dr}_1 + \mathbf{dr}_2)
\end{aligned}$$

where $\mathbf{dr} = \mathbf{r}(d\phi_1, d\phi_2) - \mathbf{r}\,(0,0)$ . If we take the limit $dt \to 0$, dividing both sides by $dt$; $\lim_{t\to 0} \frac{\mathbf{dr}}{dt} = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$. Now we have $\mathbf{dr} = \mathbf{d}\boldsymbol{\phi} \times \mathbf{r}$, and so the above show

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega}_1 \times \mathbf{r} + \boldsymbol{\omega}_2 \times \mathbf{r} \Longrightarrow \boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$

Note that $\boldsymbol{\omega}$ can be calculated in any co-ordinate system as *it is a physical vector.*

## 1.4 Rotated Reference Frames

Now we return to describing transformations of vectors between co-ordinate frames rotating with respect to one another. We will define two reference frames, the *body frame* which is fixed with a rotating body, and the *space frame* – the inertial frame.

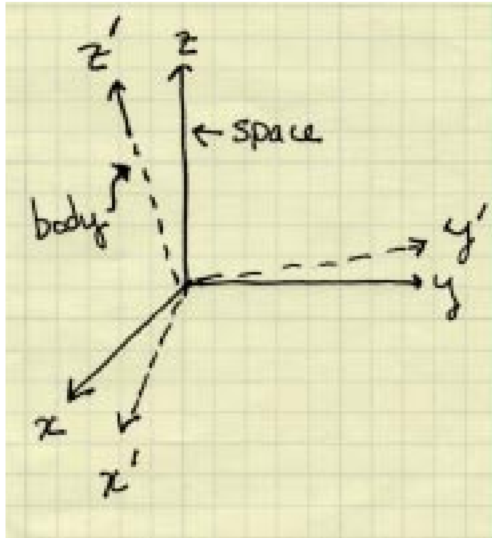
In the space frame we write

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \qquad \text{K frame}$$

and in the body (rotating) frame

$$\mathbf{r}' = \begin{pmatrix} r_1' \\ r_2' \\ r_3' \end{pmatrix} \qquad \text{K}' \text{ frame}$$

We need three numbers to specify the orientation of $K'$ wrt $K$ (a rigid body with fixed CM requires 3 quantities to specify its orientation). An arbitrary 3-D rotation can be described




bo two angles to specify the location of the body-fixed axis and 1 to specify the rotation about that axis.

We know how to transform vectors between frames:

$$\begin{aligned} a_1 &= \mathbf{i}\cdot\mathbf{a}^{'}{=}a_1^{'}\mathbf{i}\cdot\mathbf{i}' + a_2^{'}\mathbf{i}\cdot\mathbf{j}' + a_3^{'}\mathbf{i}\cdot\mathbf{k}' \\ a_2 &= \mathbf{j}\cdot\mathbf{a}'{=}\,..... \end{aligned}$$

This transformation can be represented by a $3\times 3$ matrix:

$$V \equiv \begin{pmatrix} i\cdot i' & i\cdot j' & i\cdot k' \\ j\cdot i' & j\cdot j' & j\cdot k' \\ k\cdot i' & k\cdot j' & k\cdot k' \end{pmatrix}$$

This may look strange, since we have nine entries in the matrix, but we have only 3 dof. We must therefore have six relations between the entries. We'll see soon what these are.

$U$ is the matrix that takes us from the unprimed to the primed frame:

$$\mathbf{r} = \mathbf{V}\cdot\mathbf{r}' \text{ or } r_k = V_{ki}r_i^{'}$$

Here $\mathbf{r}$ and $\mathbf{r}'$ are understood to be column vectors containing the co-ordinates of the vectors. We have to be careful, as these (the co-ordinates) relate to a particular reference frame, whereas $\mathbf{r}$ refers to the physical vector.

Repeated rotations are represented as successive linear transformations:

$$\mathbf{V}_1 \text{ followed by } \mathbf{V}_2 \text{ is represented by } \mathbf{V} = \mathbf{V}_2\mathbf{V}_1$$

The relations between the components of $V$ come from preserving the lengths of the vectors (we don't allow stretching of the coordinate systems)

$$r^2 = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = r'^2 = \tilde{\mathbf{r}}'\tilde{\mathbf{V}}\mathbf{V}\mathbf{r}^{'}$$

$$\Longrightarrow \tilde{\mathbf{V}} \cdot \mathbf{V} = \mathbf{I}$$

so

$$\mathbf{r}' = \tilde{\mathbf{V}} \cdot \mathbf{r}$$

takes us from space co–ordinates to body co-ordinates. $\tilde{\mathbf{V}}\mathbf{V} = \mathbf{I} \Longrightarrow \tilde{\mathbf{V}} = \mathbf{V}^{-1}$ so $\mathbf{V}$ is an *orthogonal* matrix. The condition that $\mathbf{V}$ be orthogonal would appear to give 9 equations between the entries, but $\tilde{\mathbf{V}}\mathbf{V}$ is automatically symmetric. A symmetric $3 \times 3$ matrix has six independent elements, so we have 6 relations between the entries, and 3 independent dof.

## 1.5 Rotating Reference Frames

Assume $K'$ (body frame) may be rotating:

$$\mathbf{V} = \mathbf{V}(t)$$

also, assume the point $P$ in $K'$ can be moving (for example a bug crawling along a turntable). We want to relate the time derivatives of $P$ in the space frame to the time derivatives in the rotating (body) system.

$\mathbf{v}|_{space} \equiv \dfrac{d\mathbf{r}}{dt}$ (time derivatives of the space coordinates of $P$ expressed in $K$ (space system)

$\mathbf{v}|'_{body} \equiv \dfrac{d\mathbf{r}'}{dt}$ (time derivatives of the body coordinates expressed in $K'$ (body system).

So we have

$$\mathbf{r} = \mathbf{V} \cdot \mathbf{r}'$$

take the time derivative of both sides:

$$\mathbf{v}|_{space} = \dot{\mathbf{V}} \cdot \mathbf{r}' + \mathbf{V} \cdot \mathbf{v}|'_{body}$$

where the first term, $\dot{\mathbf{V}} \cdot \mathbf{r}'$ is the relative rotation of the frames in time, and the second term $\mathbf{V} \cdot \mathbf{v}|'_{body}$ is the transformation of the body velocity into the space coordinate system. So since $\tilde{\mathbf{V}}\mathbf{V} = \mathbf{I}$,

$$\mathbf{v}|_{space} = \dot{\mathbf{V}}\tilde{\mathbf{V}}\mathbf{V} \cdot \mathbf{r}' + \mathbf{V} \cdot \mathbf{v}|'_{body}$$

and using $\mathbf{r} = \mathbf{V} \cdot \mathbf{r}'$

$$\mathbf{v}|_{space} = \dot{\mathbf{V}}\tilde{\mathbf{V}} \cdot \mathbf{r} + \mathbf{V} \cdot \mathbf{v}|'_{body}$$

Now the last term, $\mathbf{V} \cdot \mathbf{v}|'_{body}$ is the transformation of $\mathbf{v}|'_{body}$ into the space system ($K' \to K$). And so everything in the above equation *is expressed in the same (space) system*, and we can replace with a vector identity.

Note: $\mathbf{V} \cdot \mathbf{v}|'_{body} \equiv \mathbf{v}|_{body}$, and

$$\mathbf{v}|_{body} = \text{ body velocity resolved into space co-ordinates}$$

$$\mathbf{v}|'_{space} = \text{ space velocity resolved into body co-ordinates}$$
$$\mathbf{v}|'_{body} = \text{ body velocity resolved into body co-ordinates}$$

Now $\dot{\mathbf{V}}\tilde{\mathbf{V}}$ is an anti-symmetric matrix

$$\dot{\mathbf{V}}\tilde{\mathbf{V}} \equiv \mathbf{A}$$

(note we are working in space co-ordinates, and the elements of $\mathbf{A}$ depend on the co-ordinate system). Prove the antisymmetry:

$$\mathbf{V}\tilde{\mathbf{V}} = \mathbf{I} \rightarrow \ \dot{\mathbf{V}}\tilde{\mathbf{V}} + \mathbf{V}\frac{d\tilde{\mathbf{V}}}{dt} = \mathbf{0}$$

and

$$transpose(\dot{\mathbf{V}}\tilde{\mathbf{V}}) = \mathbf{V}\frac{d\tilde{\mathbf{V}}}{dt}$$
$$\tilde{\mathbf{A}} = transpose(\dot{\mathbf{V}}\tilde{\mathbf{V}}) = -\dot{\mathbf{V}}\tilde{\mathbf{V}} = -\mathbf{A}$$

## 1.6 Instantaneous Angular Velocity

Write $\mathbf{A}$ in the most general form of an anti-symmetric matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

Note here the $\omega$'s refer to elements of $\mathbf{A}$ calculated in the space $(K)$ system. In general, these are not equal to the components in the $K'$ system

$$\mathbf{A}' = \tilde{\mathbf{V}}\mathbf{A}\mathbf{V}$$

Where in this operation $\mathbf{V}$ converts a vector from primed to unprimed system, $\mathbf{A}$ operates on the vector, and $\tilde{\mathbf{V}}$ converts back to the primed frame. If we look at the elements of $\mathbf{A}\cdot\mathbf{r}$ we can see that

$$\mathbf{A}\cdot\mathbf{r} = \boldsymbol{\omega}\times\mathbf{r}$$

where $\omega \equiv (A_{32}, A_{13}, A_{21})$ evaluated in the $K$ frame. We can see that if $\mathbf{r} \parallel \boldsymbol{\omega}$ then the cross product is =0, and this proves that $\boldsymbol{\omega}$ defines the axis of rotation. So $\boldsymbol{\omega}$ is the *instantaneous* angular velocity (in general $\boldsymbol{\omega}$ is not constant in time).

So we can rewrite:

$$\begin{aligned} \mathbf{v}|_{space} &= \dot{\mathbf{V}}\tilde{\mathbf{V}}\mathbf{r} + \mathbf{v}|_{body} \\ &= \boldsymbol{\omega}\times\mathbf{r} + \mathbf{v}|_{body} \end{aligned}$$

Note: $\boldsymbol{\omega}$ is a physical vector, its components can be expressed either in the space or the body frame:

$$\mathbf{A}' = \tilde{\mathbf{V}}\mathbf{A}\mathbf{V} = \mathbf{A}' = \tilde{\mathbf{V}}\dot{\mathbf{V}}$$

This relation can be used to calculate the components of $\omega$ in the body system, given them in the space system.

A vector is any 3 numbers that transform under rotation like the co-ordinates of a point in space. Any vector will transform as

$$\mathbf{e} = \mathbf{V}\mathbf{e}'$$

Similarly, the time derivative an any vector will transform as

$$\frac{d}{dt}|_{space} = \left[\boldsymbol{\omega} \times \;\; + \frac{d}{dt}|_{body}\right]$$

We can apply this twice to get the acceleration transformation:

$$\mathbf{a}|_{space} = \boldsymbol{\omega} \times \boldsymbol{\omega} \times \mathbf{r} + 2\boldsymbol{\omega} \times \mathbf{v}|_{body} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \mathbf{a}|_{body}$$

## 1.7 Fictitious Forces

In an inertial frame Newton's laws dictate

$$\mathbf{F} = m\mathbf{a}_{space}$$

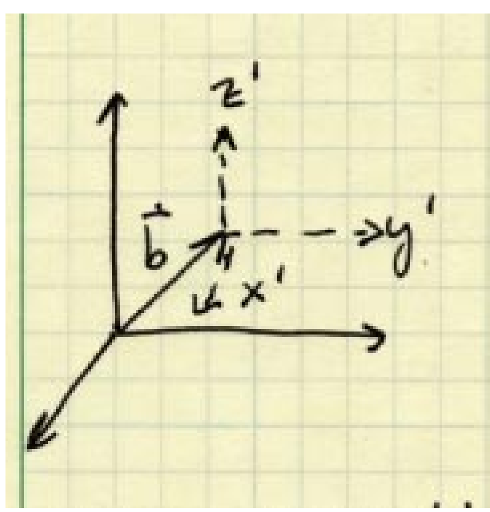
We can pretend the body system is inertial, and define an apparent force as

$$\mathbf{F}_{apparent} \equiv m\mathbf{a}_{body}$$

where

$$\mathbf{F}_{apparent} = F - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m(\boldsymbol{\omega} \times \mathbf{v}) - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$$

Here $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the centrifugal force, $2m(\boldsymbol{\omega} \times \mathbf{v})$ is the Coriolis force, and $m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ is called the Euler force.




For complete generality, lets assume the origin of $K$ and $K'$ can be accelerating wrt one another. We must add another fictitious force due to this. Let $\mathbf{b}$ be the vector connecting the origins of the two systems. Then we must add an additional term $m\frac{d^2\mathbf{b}}{dt^2}|_{space}$ to the left hand side

$$\mathbf{F}_{apparent} = \mathbf{F} - m\frac{d^2\mathbf{b}}{dt^2}|_{space} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,(\boldsymbol{\omega} \times \mathbf{v}) - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$$

### 1.7.1 Motion on the Earth's Surface

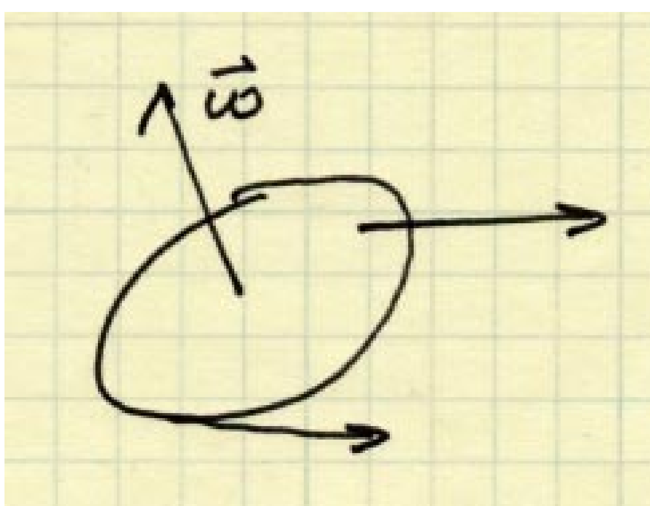
The angular velocity of the Earth is

$$\omega = \frac{2\pi}{24 \cdot 3600}\left(\frac{366.25}{365.25}\right)$$

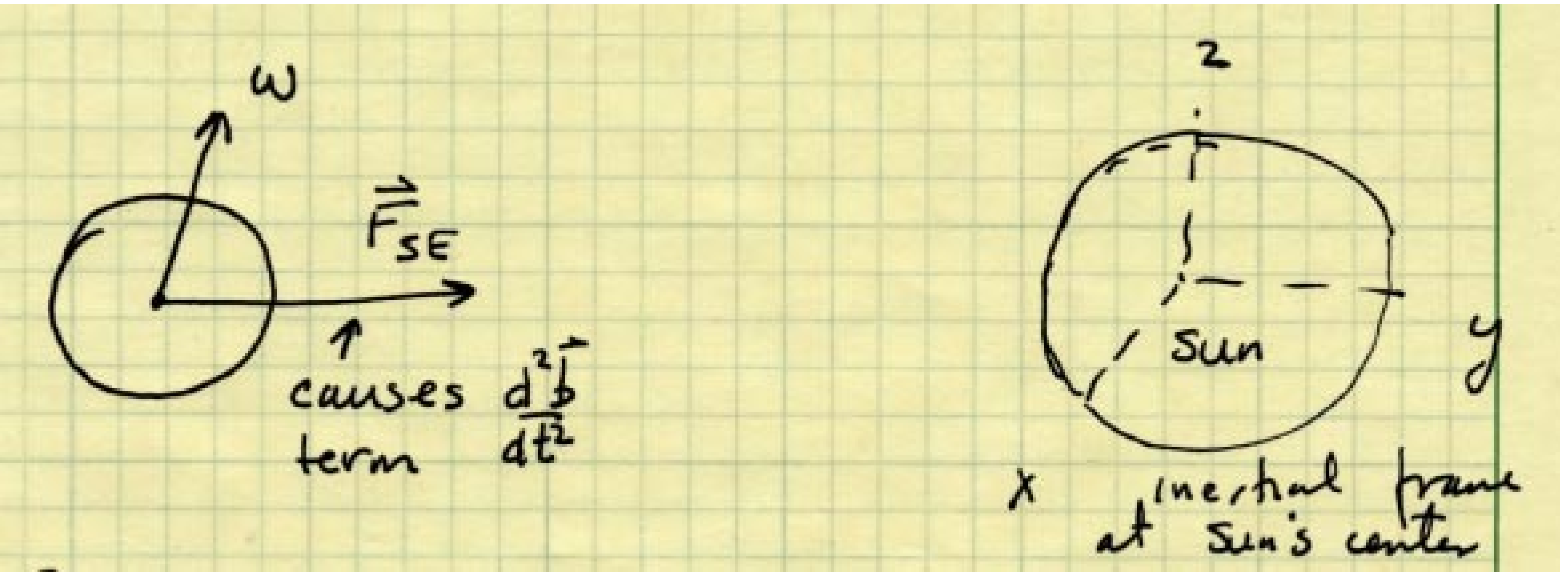
Here $\left(\frac{366.25}{365.25}\right) = \frac{\#\text{sidereal days}}{\#\text{solar days}}$ corrects for the Earth's motion around the sun. A sidereal day is the time interval for a star to return to zenith. We want the angular velocity wrt the distant stars. $\tau_{day} = 24 \cdot 3600$ is the mean time between the instants that the sun is at its highest point in the sky (noon). The Earth rotates a bit more than $2\pi$ between these instants due to the motion around the sun.

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth} = \mathbf{F} - m\left(\frac{d^2 b}{dt^2}\right)_{inertial} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{earth} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$$

Let's look at the terms one at a time. $\dot{\omega} \neq 0$ for the Earth. The axis of rotation precesses with a period of 25,000 years. The Babylonians invented astrology when the Earth's axis was pointing in a different direction wrt the stars. At that time March 21 - April 19 the sun was in Aries, but now it is in Aquarius, so we should perform the co-ordinate transformation Aries$\rightarrow$Aquarius. What causes this precession? The Earth isn't spherical. Gravitational force results in a torque, and $\omega$ precesses with a period of 25,000 years. If we compare $2\pi$ in 24 hours to $2\pi$ in 1 day, its obvious we can ignore it.




Note that the **F** in the above equation must include the gravitational forces from both Sun and Earth. The Sun also exerts a gravitational force on a particle on the Earth's surface,




however this is largely cancelled by the $-m\left(\frac{d^2\mathbf{b}}{dt^2}\right)_{inertial}$ term. The cancellation is perfect at the center of the Earth.

At the surface the residual tidal force is of order ~$\frac{R_E}{R_{SE}}$~$5\times10^{-5}$. Further, the Earth's gravitational force is much stronger than the Sun's

$$\left(\frac{M_E}{M_{sun}}\right)\left(\frac{R_{S-E}}{R_E}\right)\tilde{\ }1.7\times10^3$$

If we forget the Sun's gravity, we will be accurate to 1 part in $10^7$. So we can write

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{earth}=\mathbf{F}_g+\mathbf{F}'-2m\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{earth}-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})-m\dot{\boldsymbol{\omega}}\times\mathbf{r}$$

where $\mathbf{F}_g$ is the Earth's gravitational force on us, and $\mathbf{F}'$ represents all other forces.

Let's look at the terms in the above one by one.

### 1.7.2 Gravity

Treating the Earth as a sphere:

$$\mathbf{F}_g=-GM_Em\frac{\mathbf{r}}{r^3}$$

### 1.7.3 Centrifugal Term

The centrifugal term, $-m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$ acts on zero-velocity particles. We can look at it as altering $\mathbf{g}$ locally. Consider a particle on a scale

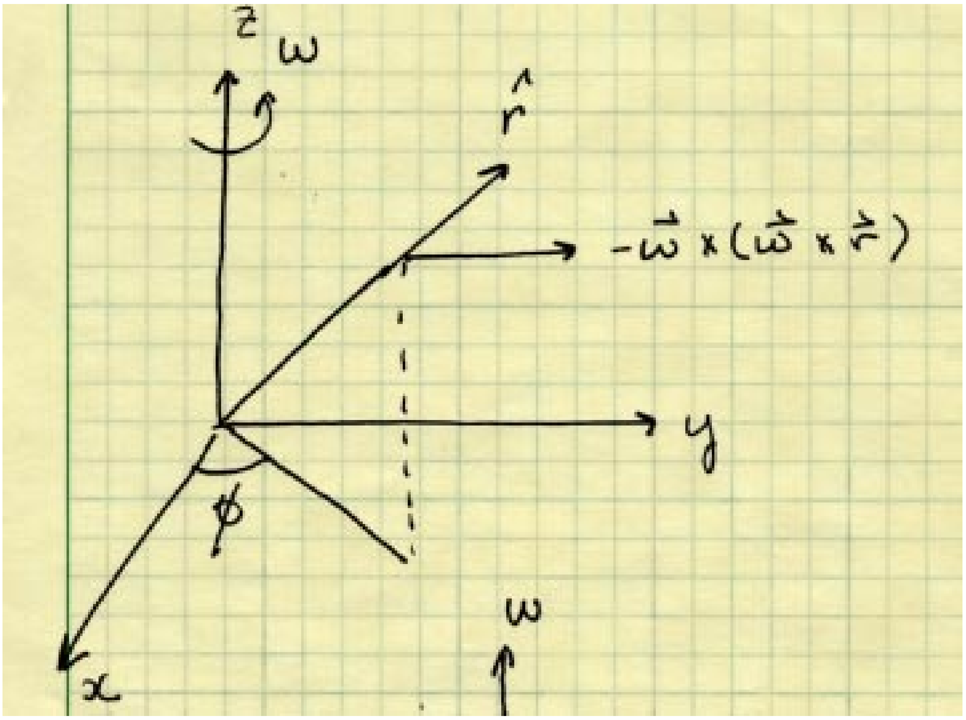
$$\left(\frac{d\mathbf{r}}{dt}\right)_{Earth}=0,\quad\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{Earth}=0$$

The scale exerts a force on the particle

$$\mathbf{F}' = -m\mathbf{g}$$

with

$$\mathbf{g} = -GM_e\frac{\mathbf{r}}{r^3} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$




$$\boldsymbol{\omega} \times \mathbf{r} = \omega r \sin\theta \ \hat{\boldsymbol{\phi}}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 r \sin\theta \left(\sin\theta \, \hat{\mathbf{r}} + \cos\theta \, \hat{\boldsymbol{\theta}}\right)$$

so

$$g = -\left(GM_e\frac{\mathbf{r}}{r^3} - \omega^2 R_e \sin^2\theta\right)\hat{\mathbf{r}} + \frac{1}{2}\sin(2\theta)\,\omega^2 R_e \hat{\boldsymbol{\theta}}$$

$\Longrightarrow$ **g** does not point toward the center of the Earth unless $\theta = 0$ or $\theta = \pi/2$.

Compare magnitudes:

$$\frac{GM_e}{R_e^2} \simeq 980 \ cm\, s^{-2}$$

$$\omega^2 R_e \simeq 3.4 \ cm \ s^{-2}$$

So $|\mathbf{g}|$ at the North pole compared to the equator is:

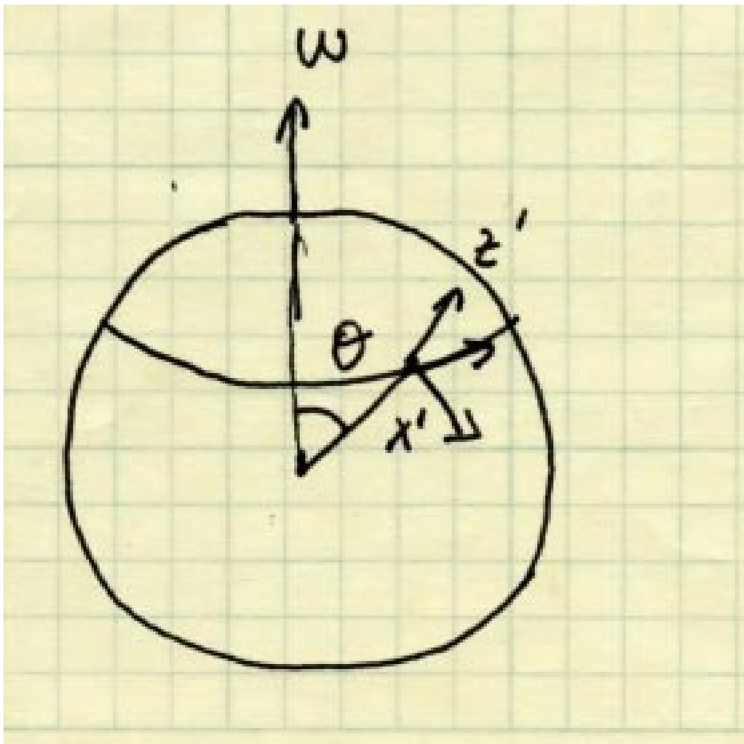
$$\Delta g = |\mathbf{g}|_{\theta=0} - |\mathbf{g}|_{\theta=\pi/2} = \omega^2 R_e \simeq 3.4 \ cm \ s^{-2}$$

The measured $\Delta g = 5.2 \ cm \ s^{-2} \rightarrow$ due to the Earth's equatorial bulge.

### 1.7.4 Coriolis Force

$$F_{coriolis} = -2m\,\boldsymbol{\omega} \times \mathbf{v}$$

The origin of this force is motion with respect to the rotation axis.




First calculate the Coriolis deflection for a particle in free fall – say a BB dropped from the top of a tower.

Particle in free fall:

$$m\left(\frac{d\mathbf{v}}{dt}\right)_{Earth} = -2m\ \boldsymbol{\omega} \times \mathbf{v} + m\mathbf{g}$$

$$\mathbf{v} = \left(\frac{d\mathbf{r}}{dt}\right)_{Earth}$$

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}$$

where $\mathbf{g}$ is the gravitational acceleration, and the second term is a small Coriolis force correction. Solve the problem approximately:

$$\mathbf{r}\,(t) = \mathbf{r}_o\,(t) + \mathbf{r}_1\,(t) + .... + h.o.t.$$

where $\mathbf{r}_o\,(t)$ is due to gravity, and $\mathbf{r}_1\,(t)$ is the first order Coriolis correction.

$$\ddot{\mathbf{r}}_o + \ddot{\mathbf{r}}_1 \simeq \mathbf{g} - 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}}_o$$

and

$$\begin{aligned}\ddot{\mathbf{r}}_o &= \mathbf{g}\\ \ddot{\mathbf{r}}_1 &= -2\boldsymbol{\omega}\times\dot{\mathbf{r}}_o\end{aligned}$$

drop from rest:

$$\dot{\mathbf{r}}_o\left(t=0\right)=0$$

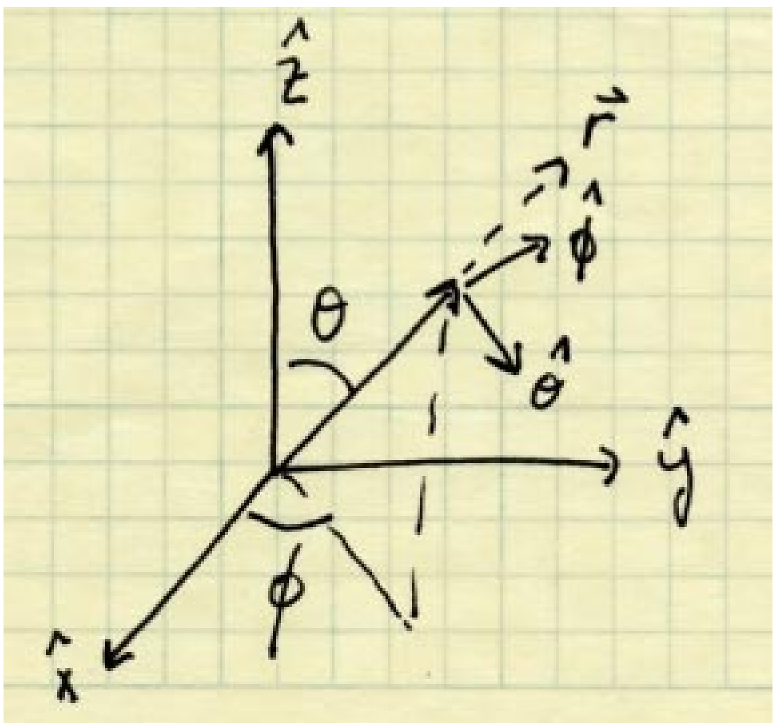
$$\mathbf{r}_o\left(t\right)=\mathbf{r}\left(0\right)+\frac{1}{2}\mathbf{g}t^2$$

$$\Longrightarrow \ddot{\mathbf{r}}_1=-2\left(\boldsymbol{\omega}\times\mathbf{g}\right)t$$

and if we integrate

$$\mathbf{r}_1\left(t\right)=-\frac{1}{3}t^3\left(\boldsymbol{\omega}\times\mathbf{g}\right)$$

Assume $g$ is constant (neglect variation with $z$)$\Longrightarrow h << R_E$




$$\begin{aligned}\boldsymbol{\omega}\times\mathbf{g} &= \omega g_r\left(\hat{\mathbf{z}}\times\mathbf{r}\right)+\omega g_\theta\left(\hat{\mathbf{z}}\times\hat{\boldsymbol{\theta}}\right)\\ &= \omega\left(g_r\sin\theta+g_\theta\cos\theta\right)\hat{\boldsymbol{\phi}}\end{aligned}$$

with:

$$g_r=-\left(\frac{GM_E}{R_E^2}-\omega^2R_E\sin^2\theta\right)$$

$$g_\theta = \frac{1}{2}\sin(2\theta)\omega^2 R_E$$

Expand using $\sin\theta = \frac{1}{2}\sin(2\theta)\cos\theta + \sin^2\theta\,\sin\theta$

$$\boldsymbol{\omega}\times\mathbf{g} = -\omega\sin\theta\left(g_o - \omega^2 R_E\right)\hat{\boldsymbol{\phi}}$$

take $\mathbf{r}(t=0) = h\hat{\mathbf{r}}$

$$\begin{aligned}\mathbf{r}(t) &= \left[h - \frac{1}{2}\left(g_o - \omega^2 R_E\sin^2\theta\right)t^2\right]\hat{\mathbf{r}} \\ &\quad + \left[\frac{1}{4}\sin(2\theta)\omega^2 R_E t^2\right]\hat{\boldsymbol{\theta}} \\ &\quad + \left[\frac{1}{3}\omega\sin\theta\left(g_o - \omega^2 R_E\right)t^3\right]\hat{\boldsymbol{\phi}}\end{aligned}$$

to first order, the free fall time is $t_o \cong \sqrt{\frac{2h}{g_o}} \Longrightarrow$ the rotation-induced deflections are of magnitude

$$\Delta r_\theta \sim \frac{1}{4}\omega^2 R_E t_o^2 = \left(\frac{\omega^2 R_E}{g_o}\right)\frac{h}{2}$$

$$\Delta r_\phi \sim \frac{1}{3}\omega g_o t^3 = (\omega t_o)\,\frac{2}{3}h$$

$\frac{\omega^2 R_E}{g_o} \sim 3.5\times 10^{-3}$ and $\omega t_o \sim 3.3\times 10^{-4}$ if $h = 100\,m$. $\hat{\boldsymbol{\phi}}$ points East - for free fall, the Coriolis deflection is always to the East.

For particles launched parallel to the surface, the Coriolis force direction switches in Northern and Southern hemispheres (but for vertical free-fall the deflection is always E).

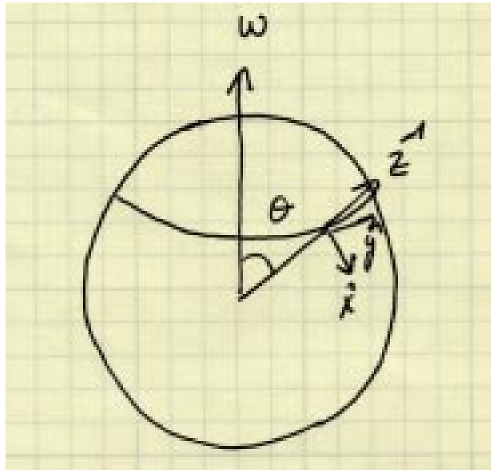
### 1.7.5 The Foucault Pendulum

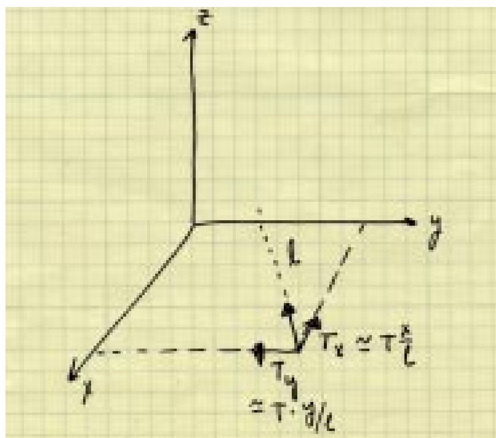
Consider a very long pendulum undergoing small amplitude oscillations – ie. consider motion in the x-y plane: $\frac{dz}{dt}\approx 0$ compared to $\dot{x}, \dot{y}$.

$$\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{body} = \mathbf{g} + \frac{\mathbf{T}}{m} - 2\boldsymbol{\omega}\times\left(\frac{d\mathbf{r}}{dt}\right)_{body}$$

We consider the pendulum to be suspended from great height, and neglect deviations of from local vertical: $\mathbf{g} = g\hat{\mathbf{z}}$

$$\begin{aligned}\boldsymbol{\omega}\times\mathbf{v} &\cong \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -\omega\sin\theta & 0 & \omega\cos\theta \\ \dot{x} & \dot{y} & 0\end{vmatrix} \\ &= -\dot{y}\omega\cos\theta\,\hat{x} + \omega\dot{x}\cos\theta\,\hat{\mathbf{y}} + -\dot{y}\omega\sin\theta\,\hat{\mathbf{z}}\end{aligned}$$




$$\left(\frac{d^2x}{dt^2}\right)_{body} = \ddot{x} \cong -\frac{T}{m}\frac{x}{l} + 2\dot{y}\omega\cos\theta$$

$$\left(\frac{d^2y}{dt^2}\right)_{body} = \ddot{y} \cong -\frac{T}{m}\frac{y}{l} - 2\dot{x}\omega\cos\theta$$

For small displacements, $T \approx mg$. Let $\alpha^2 = T/ml\,,\ \omega_z = \omega\cos\theta$

$$\begin{aligned} \ddot{x} + \alpha^2 x &\cong 2\omega_z \dot{y} \\ \ddot{y} + \alpha^2 y &\cong -2\omega_z \dot{x} \end{aligned}$$

and we have two coupled equations. Look for solutions of the form $x \sim e^{i\Omega t}$, $y \sim e^{i\Omega t}$

$$\left(-\Omega^2 + \alpha^2\right) x - 2i\omega_z \Omega y = 0$$

$$\left(-\Omega^2 + \alpha^2\right) y + 2i\omega_z \Omega x = 0$$

$$\Longrightarrow \left(-\Omega^2 + \alpha^2\right)^2 - 4\omega_z^2\Omega^2 = 0$$

$$\alpha^2 - \Omega^2 - 2\omega_z\Omega = 0$$

$$\Omega = -\omega_z \pm \sqrt{\omega_z^2 + \alpha^2}$$

The normal modes are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

add these modes to get any motion:

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{-i\omega_z t}\left(\frac{c_1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{i\alpha t} + \frac{c_2}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-i\alpha t}\right)$$

for $t = 0, y = 0 \quad x = x_m$

$$x\left(t\right) = e^{-i\omega_z t}\frac{x_m}{\sqrt{2}}\left(e^{i\alpha t} + e^{-i\alpha t}\right)$$

$$\begin{aligned} \mathrm{Re}\left(x\right) &= \sqrt{2}\cos\left(\omega_z t\right) x_m \cos\left(\alpha t\right) \\ y &= \sqrt{2}\sin\left(\omega_z t\right) x_m \sin\left(\alpha t\right) \end{aligned}$$

$\Longrightarrow$ x and y rotate with frequency $\omega_z = \omega \sin\theta$.