1 Topic 8: Rigid Body Dynamics

Reading assignment: Hand and Finch Chapter 8

In the first week of class, we showed that the total torque on a system of particles can be written

$$\mathbf{N} = \mathbf{R} \times \dot{\mathbf{P}} + \sum_\alpha \mathbf{r}_\alpha \times \dot{\mathbf{p}}_\alpha$$

we can write the term about the CM as $\dot{\mathbf{s}}$ (”spin” angular momentum), where

$$\begin{aligned} s &= \sum_\alpha m_\alpha \mathbf{r}_\alpha \times \mathbf{v}_\alpha \\ &= \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\omega \times \mathbf{r}_\alpha) \end{aligned}$$

for a rigid body. Note: ω, r_α are physical vectors whose components can be derived either in an inertial or a body-fixed frame.

Expand the triple cross product:

$$\mathbf{s} = \sum m_\alpha \left(\boldsymbol{\omega} r_\alpha^2 - \mathbf{r}_\alpha \left(\mathbf{r}_\alpha \cdot \boldsymbol{\omega} \right) \right)$$

i.e.

$$s_x = \sum_\alpha m_\alpha \omega_x \left(r_\alpha^2 - x^2 \right) - m_\alpha \omega_y y_\alpha x_\alpha - m_\alpha \omega_z x_\alpha z_\alpha$$

or

$$s_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

and similarly for the y and z components. Here we have defined

$$I_{xx} = \sum m_\alpha \left(r_\alpha^2 - x_\alpha^2 \right)$$

or in the continuous limit:

$$I_{xx} = \int_V dm \left(r^2 - x^2 \right)$$

Think of I as an operator that takes ω as an input, and produces $\mathbf{s}$ as an output. I is a real function of vector arguments which transforms one vector into another vector:

$$\mathbf{s} = \mathbf{I} \cdot \boldsymbol{\omega}$$

I is the *moment of inertia tensor* – a 2^{nd} rank tensor. A tensor is defined by how its components transform under rotation – similar to the way we defined a vector last week. Operationally, it is identical to a matrix, and we will treat it as such.

I has the Cartesian components

$$I_{ij} = \int_V dm \left(r^2 \delta_{ij} - r_i r_j \right)$$

or

$$I = \int dm \begin{pmatrix} r_2^2 + r_3^2 & -r_1 r_2 & -r_1 r_3 \\ sym. & r_1^2 + r_3^2 & -r_2 r_3 \\ sym. & sym. & r_1^2 + r_2^2 \end{pmatrix}$$

I is a *real, symmetric* tensor.

We can do something similar for the kinetic energy:

$$\begin{aligned} T &= \frac{1}{2} M \dot{\mathbf{R}}_{cm}^2 + \frac{1}{2} \sum_\alpha m_\alpha \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) \\ &= T \ \text{ of CM} \quad + \qquad T \text{ about CM} \end{aligned}$$

Rearrange this expression:

$$\begin{aligned} \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) &= \mathbf{r}_\alpha \cdot \left[\left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) \times \boldsymbol{\omega}\right] \\ &= \mathbf{r}_\alpha \cdot \left[\left(\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right) \mathbf{r}_\alpha - \left(\boldsymbol{\omega} \cdot \mathbf{r}_\alpha\right) \boldsymbol{\omega}\right] \end{aligned}$$

so

$$T_{\text{about CM}} = \frac{1}{2} \sum m_\alpha \left[\omega^2 r_\alpha^2 - \left(\boldsymbol{\omega} \cdot \mathbf{r}_\alpha\right)^2\right]$$

and for a continuous distribution:

$$\begin{aligned} T_{\text{about CM}} &= \frac{1}{2} \int_V dm \ \left[\omega^2 r_\alpha^2 - \left(\boldsymbol{\omega} \cdot \mathbf{r}\right)^2\right] \\ &= \frac{1}{2} \int_V dm \ \left[\omega_i \omega_i r^2 - \left(\omega_i r_i\right) \left(\omega_j r_j\right)\right] \\ &= \frac{1}{2} I_{ij} \omega_i \omega_j = \frac{1}{2} \tilde{\boldsymbol{\omega}} I \boldsymbol{\omega} \end{aligned}$$

Note: T is a scalar independent of the co-ordinate system used to express ω, I. It is not convenient to calculate I_{ij} in the inertial frame, since the point masses are moving – i.e. $\mathbf{r}_\alpha$ changes with time in the inertial frame, so I_{ij} also changes with time in this frame. It is much easier to work in the rotating, body-fixed frame, since I_{ij} is constant in time (but $\boldsymbol{\omega}$ is often $\boldsymbol{\omega}(t)$).

We have been considering motion with no point on the body fixed. We broke things into motion of the CM and motion about the CM, and the I_{ij} we calculated was wrt the CM (CM is the origin). This system satisfies

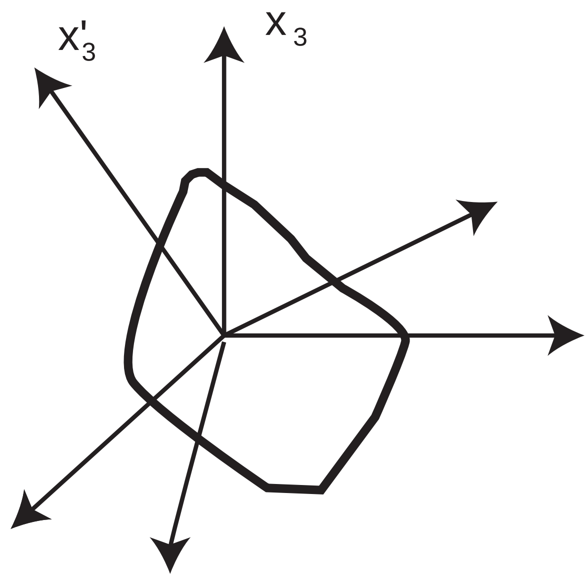
$$\int dm \ \mathbf{r} = 0 \text{ and } \int d^3\mathbf{r} \ \rho(\mathbf{r}) \, \mathbf{r} = 0$$

1.1 Motion of a Body with One Point Fixed

We want the general case with one point in the rotating body fixed.

x_i' rotates with the body

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{r}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{r}$$



but

$$\left(\frac{d\mathbf{r}}{dt}\right)_{body} = 0 \text{ for a rigid body}$$

and

$$\left(\frac{d\mathbf{r}}{dt}\right)_{inertial} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\begin{aligned} T &= \frac{1}{2}\sum_{\alpha} m_\alpha \dot{\mathbf{r}}_\alpha \cdot \dot{\mathbf{r}}_\alpha \\ &= \frac{1}{2}\sum_{\alpha} m_\alpha \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_\alpha\right) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} \end{aligned}$$

but now the origin used to calculate I_{ij} is at the fixed point, not the CM.

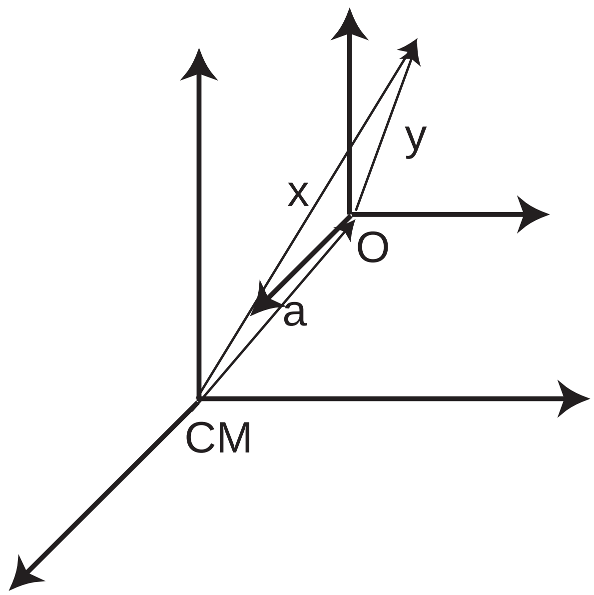
1.2 The Parallel Axis Theorem

Since I_{ij} depends on the choice of origin, and sometimes we want the CM to be the fixed point, and sometimes we want some other point in the body to be fixed it is useful to relate the two.

The easiest way to calculate I_{ij} is wrt the CM. Let $\mathbf{x}, \mathbf{y}$ be vectors to a mass point from the CM, and from the origin respectively.

$$\mathbf{x} = \mathbf{y} + \mathbf{a}$$

$$\begin{aligned} I_{ij} &= \int dm \; \left(y^2\delta_{ij} - y_i y_j\right) \quad \text{(translated system)} \\ &= \int dm \; \left[\left(x^2 - 2x \cdot a + a^2\right)\delta_{ij} - x_i x_j + a_i x_j + x_i a_j - a_i a_j\right] \end{aligned}$$



but, $\int dm\ x_i = 0$ (origin is at CM), so

$$I_{ij} = I_{ij}^{CM} + M\left(a^2\delta_{ij} - a_i a_j\right)$$

1.2.1 Example: Thin Disk

Uniform mass/unit area: $\sigma = \frac{M}{\pi R^2}$

$$M = \int_0^R 2\pi r\,\sigma\ dr$$

$$I_{zz}^{CM} = \int_0^R 2\pi r\,\sigma\left(r^2\delta_{zz} - 0\cdot 0\right)dr = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{1}{2}MR^2 \text{ (CM at origin)}$$

Now calculate with the axis at the edge of the disk:

$$I_{zz} = I_{zz}^{CM} + M\left(R^2\delta_{zz} - 0\cdot 0\right) = \frac{3}{2}MR^2$$

which is much easier than integrating directly.

1.3 Principal Axes and Principal Axis Transformation

A real, symmetric matrix, which I is, can be diagonalized by making a rotation of the coordinate system. In other words, there exist an **R** (where **R** is an *orthogonal* transformation) where

$$\mathbf{I}' = \mathbf{R}\mathbf{I}\tilde{\mathbf{R}} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

$\mathbf{RI\tilde{R}}$ is the law by which $\mathbf{I}$ (a second rank tensor) transforms under rotation. (You can prove this by considering $T = \frac{1}{2}\tilde{\boldsymbol{\omega}}\mathbf{I}\boldsymbol{\omega}$, and applying a rotation to change co-ordinates: $\boldsymbol{\omega}' = \mathbf{R}\boldsymbol{\omega}$. The kinetic energy is a scalar, and so must remain invariant to this transformation.) The real, symmetric matrix is a special case of a Hermitian matrix which we know has real eigenvalues and orthogonal eigenvectors. The key point is that an orthogonal transformation (which represents a rotation) does the diagonalization.

The matrix R which diagonalizes I is made up of the eigenvectors, and we have the eigenvalue equation:

$$\mathbf{Iv} = \lambda \mathbf{v}$$

Physically, $I_1, I_2, I_3 > 0$.

In general, we need 6 numbers to specify I: either I_1, I_2, I_3 and the orientation of the axes, or 6 elements of the general, symmetric tensor. I_1, I_2, I_3 are called the *principal moments of inertia.*

We can express the kinetic energy and the angular momentum simply in the principal axis system. In general,

$$T = \frac{1}{2} I_{ij}\omega_i\omega_j$$

where I_{ij} is normally calculated in the *body fixed* system, so the components of ω_i refer to the same body fixed system:

$$\boldsymbol{\omega} = \omega_i \hat{x}'_i$$

$\hat{x}'_i$ = unit vectors in body fixed system. Or

$$\omega_i = \hat{\mathbf{x}}'_i \cdot \boldsymbol{\omega}$$

Transform to the principal axis system:

$$\mathbf{I} = \mathbf{R}\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}\tilde{\mathbf{R}}$$

$$I_{ij} = R_{ik} I'_k R_{jk}$$

$$\begin{aligned} T &= \frac{1}{2} R_{ik} I'_k R_{jk}\omega_i\omega_j \\ &= \frac{1}{2}\omega_i R_{ik} I'_k R_{jk}\omega_j \end{aligned}$$

$$I_{ij} = R_{ik} I'_k R_{jk}$$

R is made up of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ – the unit vectors which define the principal axis system (the eigenvectors)

$$R = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \\ | & | & | \end{bmatrix}$$

The entries of this matrix are just the projections of $\mathbf{v}_i$ on the unit vectors $\hat{\mathbf{x}}'_i$ of the body-fixed co-ordinate system

$$R_{ij} = \hat{\mathbf{x}}'_i \cdot \mathbf{v}_j$$

So:

$$\begin{aligned} T &= \frac{1}{2}\omega_i R_{ik} I'_k R_{jk} \omega_j \\ &= \frac{1}{2}\left(\boldsymbol{\omega} \cdot \hat{\mathbf{x}}'_i\right)\left(\hat{\mathbf{x}}'_i \cdot \mathbf{v}_k\right) I'_k \left(\mathbf{v}_k \cdot \mathbf{x}'_j\right)\left(\hat{\mathbf{x}}'_j \cdot \boldsymbol{\omega}\right) \end{aligned}$$

but notice that $\left(\boldsymbol{\omega} \cdot \hat{\mathbf{x}}'_i\right)\left(\hat{\mathbf{x}}'_i \cdot \mathbf{v}_k\right) = \boldsymbol{\omega} \cdot \mathbf{v}_k$ (i is summed over), so that

$$T = \frac{1}{2}\left(\boldsymbol{\omega} \cdot \mathbf{v}_k\right) I'_k \left(\mathbf{v}_k \cdot \boldsymbol{\omega}\right)$$

If we redefine $\omega_1, \omega_2, \omega_3$ to represent the components of $\boldsymbol{\omega}$ *in the principal axis system*

$$\boldsymbol{\omega}_1 = \mathbf{v}_1 \cdot \boldsymbol{\omega}, \;\; \boldsymbol{\omega}_2 = \mathbf{v}_2 \cdot \boldsymbol{\omega}, \;\; \boldsymbol{\omega}_3 = \mathbf{v}_3 \cdot \boldsymbol{\omega}$$

then:

$$T = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

This is the standard form for the rotational kinetic energy of a rigid body. Remember $\omega_1, \omega_2, \omega_3$ are the components of $\boldsymbol{\omega}$ *in the principal axis system*; the special body-fixed system that diagonalizes I. The principal axes are usually symmetry axes of the body (if they exist), and can often be recognized without calculation.

1.3.1 L in the Principal Axis System

The angular momentum is

$$L_i = I_{ij}\omega_j$$

(L is in the body-fixed co-ordinate system, as $\boldsymbol{\omega}$ is)

$$L_i = \mathbf{L} \cdot \hat{\mathbf{x}}'_i \qquad \omega_i = \hat{\mathbf{x}}_i \cdot \boldsymbol{\omega}$$

so

$$\mathbf{L} = \hat{\mathbf{x}}'_i I_{ij}\left(\hat{\mathbf{x}}'_j \cdot \boldsymbol{\omega}\right)$$

again, we write

$$\begin{aligned} I_{ij} &= R_{ik} I'_k R_{jk} \\ &= \left(\hat{\mathbf{x}}'_i \cdot \mathbf{v}_k\right) I'_k \left(\mathbf{v}_k \cdot \hat{\mathbf{x}}'_j\right) \end{aligned}$$

and so

$$\begin{aligned} \mathbf{L} &= \hat{\mathbf{x}}'_i \left[\left(\hat{\mathbf{x}}'_i \cdot \mathbf{v}_k\right) I'_k \left(\mathbf{v}_k \cdot \hat{\mathbf{x}}'_j\right)\left(\hat{\mathbf{x}}'_j \cdot \boldsymbol{\omega}\right)\right] \\ &= \mathbf{v}_k I'_k \left(\mathbf{v}_k \cdot \boldsymbol{\omega}\right) \end{aligned}$$

and

$$\mathbf{v}_k \cdot \mathbf{L} = I'_k \left(\mathbf{v}_k \cdot \boldsymbol{\omega}\right)$$

so if L_1, L_2, L_3 and $\omega_1, \omega_2, \omega_3$ are the components of L and ω in the principal axis system, then

$$L_1 = I_1\omega_1, \quad L_2 = I_2\omega_2, \quad L_3 = I_3\omega_3$$

In the principal axis system.

1.4 Rigid Body Equations of Motion

In an inertial frame,

$$\left(\frac{d\mathbf{L}}{dt}\right)_{inertial} = \mathbf{N}$$

where N is the torque:

$$\mathbf{N} = \sum_{\alpha} \mathbf{r}_\alpha \times \mathbf{F}_\alpha$$

The inertial frame can have the origin at a fixed point of the rotating body, or the CM depending on circumstance.

Since the relationship between $\mathbf{L}$ and $\boldsymbol{\omega}$ is only simple in the body-fixed frame, rewrite as

$$\left(\frac{d\mathbf{L}}{dt}\right)_{inertial} = \left(\frac{d\mathbf{L}}{dt}\right)_{body} + \boldsymbol{\omega} \times \mathbf{L}$$

Take the body-fixed frame to be the principal axis system:

$$\mathbf{L} = L_i \mathbf{v}_i, \quad L_i = \mathbf{L} \cdot \mathbf{v}_i$$

then

$$\left(\frac{d\mathbf{L}}{dt}\right)_{body} = \frac{dL_i}{dt}\mathbf{v}_i$$

so

$$\frac{dL_i}{dt} + \varepsilon_{ijk}\omega_j L_k = N_i$$

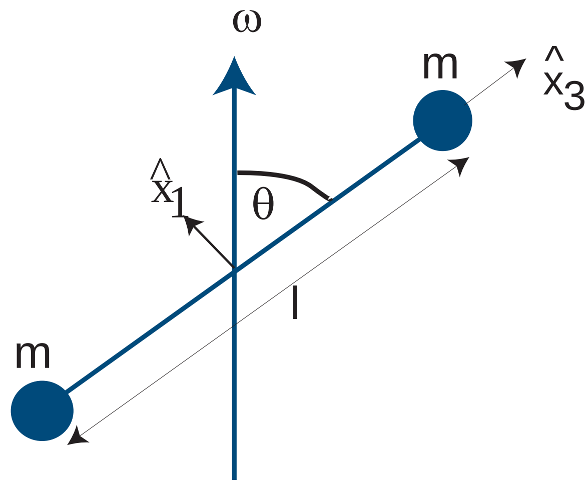
where $N_i = (\mathbf{v}_i \cdot \mathbf{N})$ = components of $\mathbf{N}$ in the body-fixed (principal axis) frame. Now ω_j are the components of ω in the principal axis frame, $\omega_j = (\mathbf{v}_j \cdot \boldsymbol{\omega})$. Using $L_i = I_i\omega_i$, we get the *Euler equations*

$$I_i \frac{d\omega_i}{dt} + \varepsilon_{ijk}\omega_j\omega_k I_k = N_i$$

$$\begin{aligned}
I_1\dot{\omega}_1 - \omega_2\omega_3\left(I_2 - I_3\right) &= N_1 \\
I_2\dot{\omega}_2 - \omega_3\omega_1\left(I_3 - I_1\right) &= N_2 \\
I_3\dot{\omega}_3 - \omega_1\omega_2\left(I_1 - I_2\right) &= N_3
\end{aligned}$$

Note: If we have an object rotating at constant angular velocity, only rotation *about a principal axis* is torque-free.

1.4.1 Example:



A dumbbell rotating: $\hat{\mathbf{x}}_1$ in the plane of $\boldsymbol{\omega}, \hat{\mathbf{x}}_3$.

$$\begin{aligned} \omega_1 &= \omega\sin\theta,\ \ \omega_2 = 0,\ \ \omega_3 = \omega\cos\theta \\ N_1 &= 0,\ \ N_2 = \omega^2\sin\theta\cos\theta\ I_1,\ \ N_3 = 0 \end{aligned}$$

where

$$I_1 = I_2 = \frac{1}{2}ml^2, \quad I_3 = 0$$

so need torque $\mathbf{N} = N\hat{\mathbf{x}}_2$ to keep the dumbbell rotating at constant ω. But, to rotate about a principal axis, $\mathbf{N} = 0$. Other examples: balancing car wheels – static balancing means CM is on the axle, dynamic balancing means $\mathbf{L}$ is along an axle.

1.5 The Force-Free Symmetric Top

Take a top where $I_1 = I_2 \neq I_3$ (ie. Earth) in the absence of external torques, $N = 0$. Let $I_1 = I_2 = I$. The Euler equations are:

$$\dot{\omega}_1 + \frac{I_3 - I}{I}\omega_2\omega_3 = 0$$

$$\dot{\omega}_2 - \frac{I_3 - I}{I}\omega_1\omega_3 = 0$$

$$\dot{\omega}_3 = 0$$

The sol'n: $\omega_3 =$ const. Let

$$\Omega = \omega_3\left(\frac{I_3 - I}{I}\right)$$

$$\dot{\omega}_1 + \Omega\omega_2 = 0$$

$$\dot{\omega}_2 - \Omega\omega_1 = 0$$

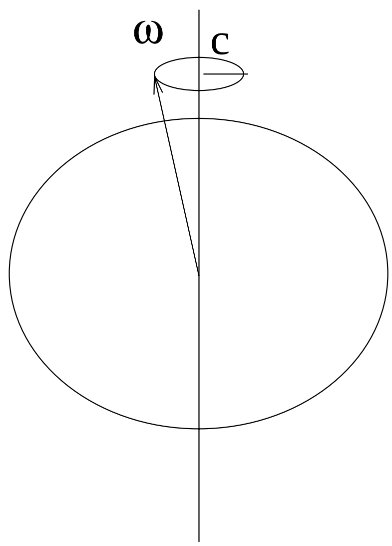
solutions will be of form $\omega_1 = a_1 e^{i\Omega t}$, $\omega_2 = a_2 e^{i\Omega t}$. Plug into eqn's of motion and we get

$$\frac{a_1}{a_2} = -i$$

$\rightarrow$ out of phase by π.

$$\begin{aligned} \omega_1 &= c\cos(\Omega t + \phi) \\ \omega_2 &= c\sin(\Omega t + \phi) \end{aligned}$$

so ω precesses around the symmetry axis $(\hat{x}'_3)$ symmetry axis with angular frequency Ω *in the body fixed frame.* i.e. for Earth



Note that the precession frequency depends on $\frac{I_3 - I}{I}$, so if the object is very nearly spherical, precession is very slow compared to ω_3. For Earth, $\omega_3 \approx \omega$ and $\frac{2\pi}{\omega_3} = 1$ day

$$\frac{I}{I_3 - I} \approx 300$$

The North pole precesses around the symmetry axis with $\tau \sim 300$ d. This is called *Chandler Wobble.* What is observed is $\tau = 440$ days with a maximum excursion of 10 meters around the principal axis. The Discrepancy is due to the fact that Earth is not rigid, but has tides. The damping timescale should be 10 - 20 years. The wobble is re-energized by earthquakes deep in the Earth, however the mechanism is still not completely understood. The liquid inner core was recently discovered to rotate at a different speed than the mantle, which may drive the wobble. Remember this was analyzed for an observer in the body-fixed frame (on Earth).

1.5.1 Motion Analyzed in the Inertial Frame

In the inertial frame, $\mathbf{L}$ is fixed. $\hat{x}, \hat{y}$, and $\hat{z}$ are the principal axes:

$$\mathbf{L} = I\left(\omega_1 \hat{\mathbf{x}} + \omega_2 \hat{\mathbf{y}}\right) + I_3 \omega_3 \hat{\mathbf{z}}$$

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{x}} + \omega_2 \hat{\mathbf{y}} + \omega_3 \hat{\mathbf{z}}$$

combine these:

$$\omega = \frac{L}{I}\hat{L} - \Omega \hat{z}$$

$(\Omega = \frac{I_3 - I}{I}\omega_3) \rightarrow \boldsymbol{\omega}, \mathbf{L}, \hat{\mathbf{z}}$ (the symmetry axis) all lie in a plane – the $\boldsymbol{\omega} - \hat{\mathbf{z}}$ plane rotates (precesses) around the fixed direction of $\mathbf{L}$. Break $\mathbf{L}$ into components along and normal to the symmetry axis

$$\mathbf{L} = I\omega_n \hat{\mathbf{n}} + I_3 \omega_3 \hat{\mathbf{z}}$$

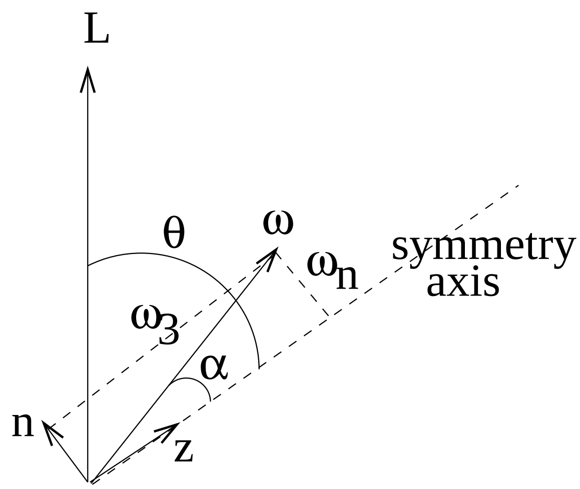
$$\omega_n \hat{\mathbf{n}} \equiv \omega_1 \hat{\mathbf{x}} + \omega_2 \hat{\mathbf{y}}$$

Look at the constants L^2, T

$$L^2 = I^2 \omega_n^2 + I_3^2 \omega_3 = const$$

$$2T = \mathbf{L} \cdot \boldsymbol{\omega} = I\omega_n^2 + I_3 \omega_3^2 = const$$

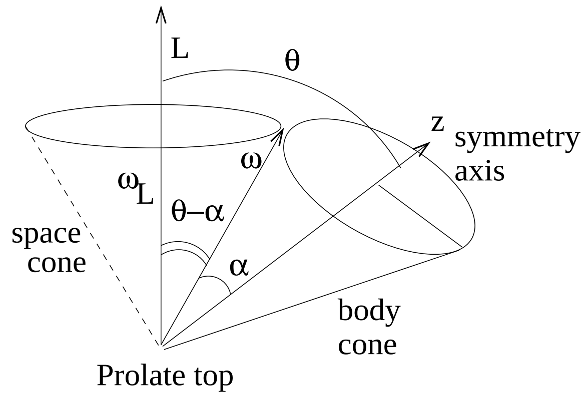
so L^2, T being constant implies ω_n, ω_3 are constant (which we already knew).



$\tan\alpha = \frac{\omega_n}{\omega_3}$ (fixed), $\tan\theta = \frac{\dot{L}n}{L_3} = \frac{I\omega_n}{I_3\omega_3} \rightarrow \tan\alpha = \frac{I_3}{I}\tan\theta$

So, the relative orientation of $\mathbf{L}, \boldsymbol{\omega}, \hat{\mathbf{z}}$ is fixed. For an oblate top (pancake like) $I_3 > I, \alpha > \theta$, and for a prolate top (cigar like) $\alpha < \theta$.

$\boldsymbol{\omega}$ sweeps out a space cone as the $\boldsymbol{\omega} - \hat{\mathbf{z}}$ plane precesses around $\mathbf{L}$. In the body frame, $\boldsymbol{\omega}$ sweeps out a cone around $\hat{\mathbf{z}}$ (body cone). For an oblate top, the space cone rolls around inside the body cone. For a prolate top, the body cone rolls on the space cone without slipping.



1.5.2 Stability of Axes. The Asymmetric Force-Free Top

Consider unequal moments of inertia and no external forces ($\mathbf{N} = 0$). The Euler equations are

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\,\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_2)\,\omega_1\omega_3 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\,\omega_2\omega_1 &= 0 \end{aligned}$$

let

$$r_1 = \frac{I_3 - I_2}{I_1},\ r_2 = \frac{I_3 - I_1}{I_2},\ r_3 = \frac{I_2 - I_1}{I_3}$$

Note: $r_{1,2,3} > 0$.

$$\begin{aligned} \dot{\omega}_1 + r_1\omega_2\omega_3 &= 0 \\ \dot{\omega}_2 - r_2\omega_3\omega_1 &= 0 \\ \dot{\omega}_3 + r_3\omega_1\omega_2 &= 0 \end{aligned}$$

We want to investigate the stability of spinning around a principal axis. First, consider rotation around axis 1 (smallest I)

$$\omega = \omega_1\hat{x}, \quad \omega_2, \omega_3 << \omega_1$$

$\omega_2\omega_3 \approx 0 \rightarrow \omega_1 \approx const.$

$$\begin{aligned} \dot{\omega}_2 - r_2\omega_1\omega_3 &= 0 \\ \dot{\omega}_3 + r_3\omega_1\omega_2 &= 0 \end{aligned}$$

solutions of form $\omega_2 = a_2 e^{i\lambda t},\ \omega_3 = a_3 e^{i\lambda t}$

$$\begin{aligned} i\lambda a_2 - r_2\omega_1 a_3 &= 0 \\ i\lambda a_3 + r_3\omega_1 a_2 &= 0 \end{aligned}$$

$$\begin{vmatrix} i\lambda & -r_2\omega_1 \\ r_3\omega_1 & i\lambda \end{vmatrix} = 0 \rightarrow -\lambda^2 + r_2 r_3 \omega_1^2 = 0$$

$$\lambda = \pm\sqrt{r_2 r_3}\omega_1, \quad \frac{a_2}{a_3} = \pm i\sqrt{\frac{r_2}{r_3}}$$

so,

$$\begin{aligned} \omega_2 &= a\sin\left(\omega_1\sqrt{r_2 r_3}t + \phi\right) \\ \omega_3 &= a\sqrt{\frac{r_3}{r_2}}\cos\left(\omega_1\sqrt{r_2 r_3}t + \phi\right) \end{aligned}$$

$\rightarrow \omega_2, \omega_3$ are bounded, and rotate around ω_1, so the motion is stable. Likewise, the solution for rotation around $\hat{\mathbf{x}}_3$ is stable. Note, however, that the Euler equations are asymmetric. Look at what this means for motion around the intermediate axis:

$$\omega \approx \omega_2 = const, \quad \omega_1, \omega_3 << \omega_2, \quad \omega_1\omega_3 = 0$$

$$\begin{aligned} \dot{\omega}_1 + (r_1\omega_2)\,\omega_3 &= 0 \\ \dot{\omega}_3 + (r_3\omega_2)\,\omega_1 &= 0 \end{aligned}$$

Get non-trivial solutions only if

$$\begin{vmatrix} i\lambda & r_1\omega_2 \\ r_3\omega_2 & i\lambda \end{vmatrix} = 0$$

$$-\lambda^2 - r_3 r_1 \omega_2^2 = 0$$

$$\lambda = \pm i\sqrt{r_3 r_1}\omega_2$$

The solution is:

$$\begin{aligned} \omega_1 &= a_1 e^{\mp\sqrt{r_3 r_1}\omega_2 t} \\ \omega_2 &= a_2 e^{\mp\sqrt{r_3 r_1}\omega_2 t} \end{aligned}$$

ie. we get exponential growth – unstable motion. Note this solution is only good for $\omega \approx \omega_2, \quad \omega_{1,3} \approx 0$. Increasing the exponential term will make $\omega_{1,3}$ large even if they started out small.

This has consequences for the design of spin-stabilized spacecraft. Rotation about the largest and the smallest principal axes are both stable if there are no dissipative forces (ie. aerodrag, non-rigidity of the body, etc.). However, spin-stabilized S/C are pancake shaped, since for rotation about this axis T is minimum for a given L ($T = \frac{L^2}{2I_1}$), and drag will decrease T while leaving L unchanged. If the S/C starts spinning about the long axis, decreasing T will change the spin to be about the axis where T is least for a given L.

1.6 Euler Angles

The next step is to consider a symmetric top in a graviational field. To do this, we will take the Lagrangian approach. We therefore must figure out the right set of generalized coordinates. There are several options – the most commonly used are called the *Euler Angles*.

We need 3 coordinates to describe a general rotation (3 angles can specify the relative orientation of any two coordinate systems with a common origin).

1.6.1 Definition of the Euler Angles

The body and space systems start out aligned – go through a series of three rotations:

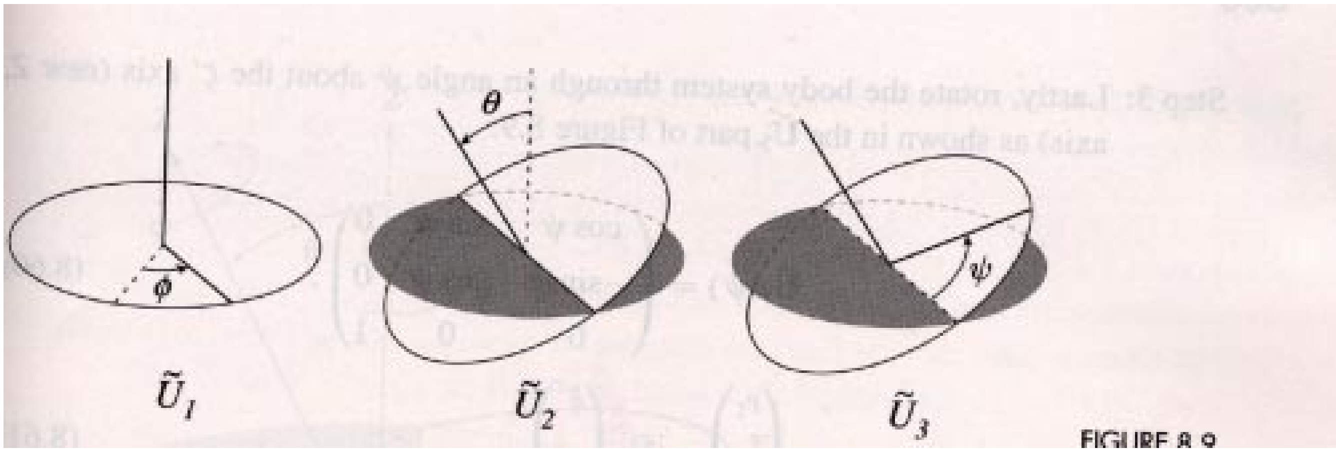


FIGURE 8.9

We can describe any arbitrary orientation by

$$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_3 \tilde{\mathbf{V}}_2 \tilde{\mathbf{V}}_1$$

$$\begin{aligned} \mathbf{r} \text{ (space co-ords)} &= \mathbf{V}\mathbf{r}' \text{ (body co-ords)} \\ \mathbf{r}' \text{ (body co-ords)} &= \tilde{\mathbf{V}}\mathbf{r} \text{ (space co-ords)} \end{aligned}$$

$$\tilde{\mathbf{V}}_1 = \begin{matrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\tilde{\mathbf{V}}_2 = \begin{matrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{matrix}$$

$$\tilde{\mathbf{V}}_3 = \begin{matrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{matrix}$$

$\tilde{\mathbf{V}} = \tilde{\mathbf{V}}_3 \tilde{\mathbf{V}}_2 \tilde{\mathbf{V}}_1, \mathbf{V} = \mathbf{V}_1 \mathbf{V}_2 \mathbf{V}_3$

$$\mathbf{V} = \begin{matrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & \sin\theta\sin\phi \\ \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{matrix}$$

The physical interpretation of $\mathbf{V}$: Any orthogonal transformation such as $\mathbf{V}$ can be represented as a rotation through an angle about an axis which is left unchanged by $\mathbf{V} \rightarrow \mathbf{V}$ has an eigenvalue = 1 and the eigenvector corresponding gives the direction of the axis. The rotation angle, Φ about this axis is a function of the three Euler angles.

Transform to a new co-ordinate system where the axis of rotation is the new z' axis $\rightarrow$ the transformation $\mathbf{V}$ in this system will take the form of a rotation Φ about this new z axis:

$$\mathbf{V}'=\mathbf{C}\mathbf{V}\tilde{\mathbf{C}}=\begin{pmatrix}\cos\Phi & \sin\Phi & 0\\ -\sin\Phi & \cos\Phi & 0\\ 0 & 0 & 1\end{pmatrix}$$

where $\mathbf{C},\tilde{\mathbf{C}}$ transform to and from the new system.

Now the trace of a matrix remains unchanged under change of co-ordinates

$$Trace\left[\mathbf{V}'\right]=Trace\left[\mathbf{C}\mathbf{V}\tilde{\mathbf{C}}\right]=Trace\left[\mathbf{V}\right]$$

so

$$Trace\left[\mathbf{V}'\right]=1+2\cos\Phi=Trace\left[\mathbf{V}\right]=\cos\theta+\cos\left(\phi+\psi\right)\left(1+\cos\theta\right)$$

and

$$\cos\left(\frac{\Phi}{2}\right)=\cos\left(\frac{\phi+\psi}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

1.6.2 Finding the Angular Velocity

The angular velocity in space co-ordinates can be read off from the elements of the antisymmetric matrix $\mathbf{A}=\dot{\mathbf{V}}\tilde{\mathbf{V}}$ in body co-ordinates, the same angular velocity can be found from looking at the elements of $\mathbf{A}=\tilde{\mathbf{V}}\dot{\mathbf{V}}$. By a straightforward (but laborious) calculation:

$$\boldsymbol{\omega}=\begin{pmatrix}\dot{\theta}\cos\psi+\dot{\phi}\sin\psi\sin\theta\\ -\dot{\theta}\sin\psi+\dot{\phi}\cos\psi\sin\theta\\ \dot{\psi}+\dot{\phi}\cos\theta\end{pmatrix}\quad\text{(body co-ordinates)}$$

$$\boldsymbol{\omega}=\begin{pmatrix}\dot{\theta}\cos\phi+\dot{\psi}\sin\phi\sin\theta\\ \dot{\theta}\sin\phi-\dot{\psi}\cos\phi\sin\theta\\ \dot{\phi}+\dot{\psi}\cos\theta\end{pmatrix}$$

From $\boldsymbol{\omega}$, we can get T, and therefore L in terms of the generalized co-ordinates θ,ψ,ϕ.

1.7 The Heavy Symmetric Top

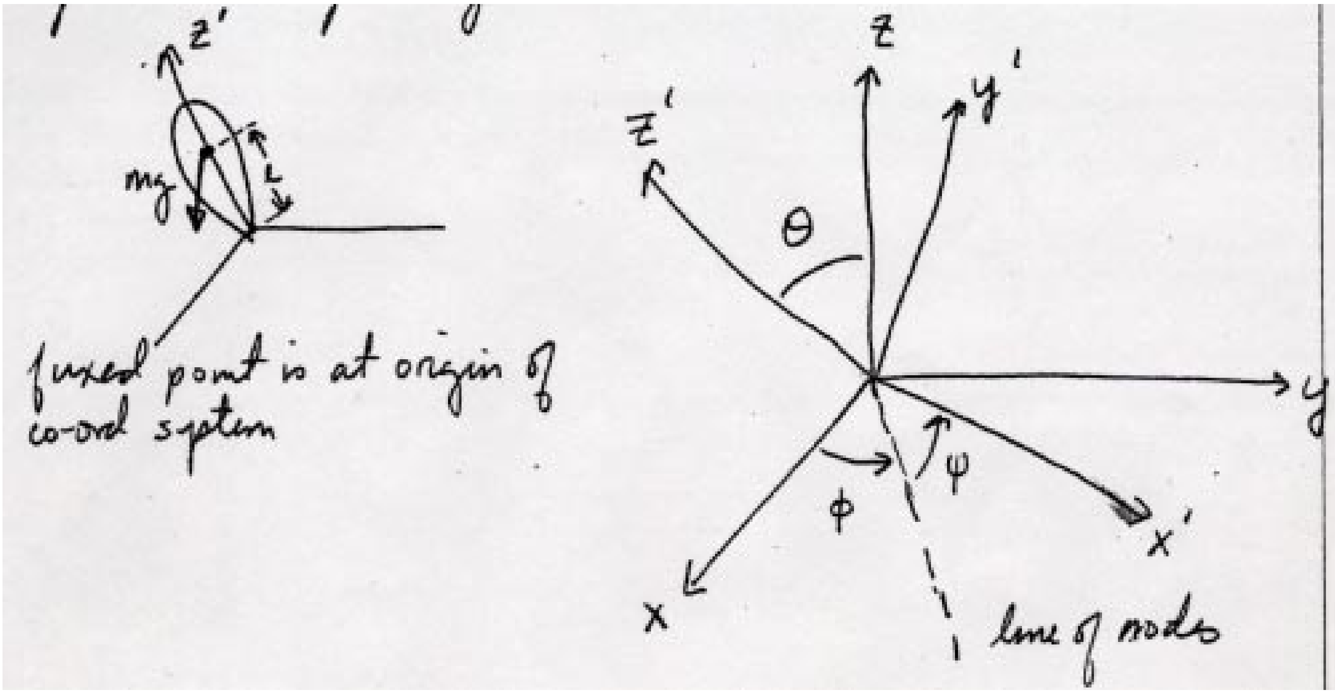
We want to consider the comtion of a rapidly spinning top in a graviational field. To analyze the motion we will take the Lagrangian approach and use the Euler angles as our generalized co-ordinates.

intertial frame – unprimed

body frome – primed

The top spins around z'.

Note: we could have used Lagrange approach to solve the force-free top. We would then get $\theta(t),\phi(t),\psi(t)$ – in other words a complete description in the body and inertial frame.



We of course compute the inertia in body co-ordinates

$$\mathbf{I}^b = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

where z' is the symmetry axis. We want to get $\boldsymbol{\omega}$ in the body co-ordinates so we can get the kinetic energy:

$\dot{\psi}$ corresponds to rotation about z' (spin)

$\dot{\phi}$ corresponds to precession about z

$\dot{\theta}$ corresponds to tip

By inspection, we can see that

$$\omega^{fixed} = \begin{pmatrix} \dot{\theta}\cos\phi + \dot{\psi}\sin\theta\sin\phi \\ \dot{\theta}\sin\phi - \dot{\psi}\sin\theta\cos\phi \\ \dot{\psi}\cos\theta + \dot{\phi} \end{pmatrix}$$

Now we want to get ω in the body frame. Resolve into ω_z^{body} and $\omega_\perp^{body}$.

$$\omega_{z'}^{body} = \dot{\psi} + \dot{\phi}\cos\theta \qquad \text{(by inspection)}$$

Now $T = \frac{1}{2}I_1\omega_\perp^{b\,2} + \frac{1}{2}I_3\omega_{z'}^{b\,2}$. So we need to find $\omega_\perp$.

$$\begin{aligned} \omega^2 &= \omega_x^{f\ 2} + \omega_y^{f\ 2} + \omega_z^{f\ 2} \\ &= \dot{\psi}^2\sin^2\theta + \dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2\cos^2\theta + 2\dot{\phi}\dot{\psi}\cos^2\theta \\ &= \dot{\psi}^2 + \dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta \end{aligned}$$

now

$$\omega^b_{z'}{}^2 = \dot{\psi}^2 + \dot{\phi}^2 \cos^2\theta + 2\dot{\psi}\dot{\phi}\cos\theta$$

so we can get $\omega_\perp^2$ from the fact that $\omega_\perp^2 = \omega^2 - \omega_{z'}^2 = \dot{\phi}^2 \sin^2\theta + \dot{\theta}^2$

So the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} I_1 \omega_\perp^2 + \frac{1}{2} I_3 \omega_{z'}^2 \\ &= \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\psi}^2 + \dot{\phi}^2 \cos^2\theta + 2\dot{\psi}\dot{\phi}\cos\theta\right) \\ &= \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 \end{aligned}$$

The Lagrangian then is:

$$L = \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Mgl\cos\theta$$

Note: The Lagrangian is cyclic in ϕ and ψ, so we have two conserved quantities:

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)\dot{\phi} + I_3 \dot{\psi}\cos\theta = const$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3 \left(\dot{\psi} + \dot{\phi}\cos\theta\right) = const$$

These correspond to the angular momentum components associated with rotation about $\mathbf{z}'$ $(\dot{\psi})$ – the body symmetry axis – and $\mathbf{z}$ (ψ) – the vertical axis.

Why are these quantities constant? The torque $-\hat{\mathbf{z}}' \times mg\hat{\mathbf{z}}$ is perpendicular to $\hat{\mathbf{z}}'$ and $\hat{\mathbf{z}}$ and lies along the line of nodes, so the associated components of $\mathbf{L}, \mathbf{L}_{\hat{z}'}$ and $\mathbf{L}_{\hat{z}}$ are constants of the motion.

We will use the expressions for p_ϕ and p_ψ to get $\dot{\psi}$ and $\dot{\phi}$ in terms of θ:

$$\dot{\psi} = \frac{p_\psi - I_3\dot{\phi}\cos\theta}{I_3}$$

substitute into the expression for p_ϕ

$$\begin{aligned} p_\phi &= \left(I_1 \sin^2\theta + I_3 \cos^2\theta\right)\dot{\phi} + I_3\dot{\psi}\cos\theta \\ &= \left(I_1 \sin^2\theta\right)\dot{\phi} + p_\psi \cos\theta \end{aligned}$$

so

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1 \sin^2\theta}$$

and

$$\dot{\psi} = \frac{P_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta}$$

could now get $\theta(t)$ from the Euler-Lagrange equns, and we would have the full solution, since we have $\dot{\psi}(\theta), \dot{\phi}(\theta)$, but to understand the general motion, it is convenient to write the energy integral

$$E = const = \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + \frac{1}{2} I_3 \omega_{z'}^{b\,2} + Mgl\cos\theta$$

now

$$\omega_{z'}^b = \dot{\psi} + \dot{\phi}\cos\theta$$
$$\rightarrow p_\psi = I_3 \omega_{z'}^b$$
$$I_3 \omega_{z'}^{b\,2} = \frac{p_\psi^2}{I_3} = const$$

so we define a new constant

$$\begin{aligned} E' &= E - \frac{1}{2} I_3 \omega_{z'}^{b\,2} = const \\ &= \frac{1}{2} I_1 \left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2 \right) + Mgl\cos\theta = const \end{aligned}$$

substitute expression for $\dot{\phi}(\theta)$:

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgl\cos\theta$$

We can look at this as a 1-D equation for motion in θ similar to what we did for central force motion:

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V_{eff}$$

where the effective potential is

$$V_{eff} = \frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + Mgl\cos\theta$$

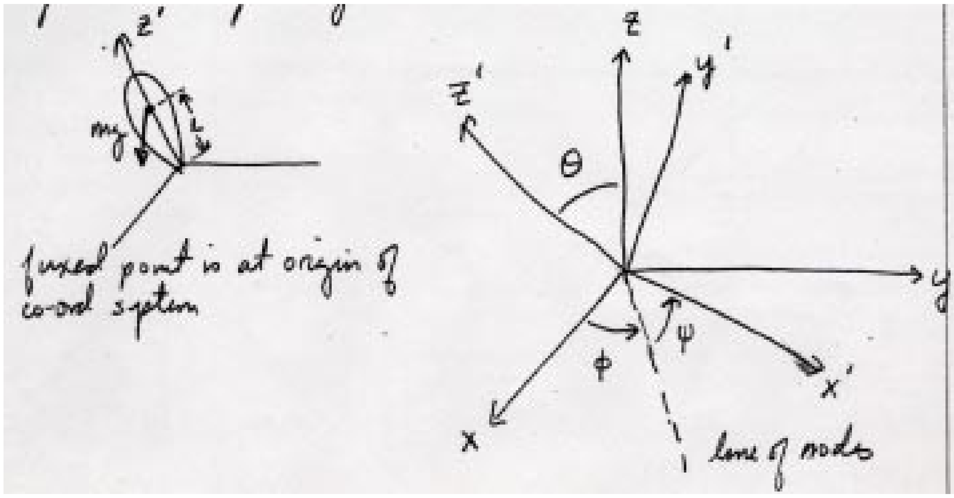
we can solve this equation to get $t(\theta)$:

$$t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}\left[E' - V_{eff}\right]}}$$

we can (formally) invert this to get $\theta(t)$, and plug into expressions for $\dot{\phi}$ and $\dot{\psi}$ to get $\phi(t), \psi(t) \rightarrow$ complete solution to the problem. We can get the qualitative features by examining V_{eff} and other equations: Plot V_{eff} from $0 \le \theta \le \pi$

From the form of V_{eff}, we see that for arbitrary E_1', the motion is limited by two extremes of θ; θ_1 and θ_2. These correspond to turning points – and are roots of the denominator of $t(\theta)$

For $E' = E_2' = V_{\min}$, θ has a single value of θ_o, and the motion is steady precession at a fixed angle of inclination.



We can get θ_o :

$$\begin{aligned}\frac{\partial V_{eff}}{\partial \theta}|_{\theta=\theta_o} &= 0 \\ &= \frac{-\cos\theta_o\left(p_\phi - p_\psi\cos\theta_o\right)^2 + p_\psi\sin^2\theta_o\left(p_\phi - p_\psi\cos\theta_o\right)}{I_1\sin^3\theta_o} - Mgl\sin\theta_o\end{aligned}$$

let $\beta = p_\phi - p_\psi\cos\theta_o$

$$\cos\theta_o\beta^2 - \left(p_\psi\sin^2\theta_o\right)\beta + MglI_1\sin^4\theta_o = 0$$

this is a quadratic in β :

$$\beta = \frac{p_\psi\sin^2\theta_o}{2\cos\theta_o}\left[1 \pm \sqrt{1 - \frac{4MglI_1\cos\theta_o}{p_\psi^2}}\right]$$

β must be real, so $\surd$ must be ≥ 0. If $\theta_o < \frac{\pi}{2}$

$$p_\psi^2 \geq 4MglI_1\cos\theta_o$$

and using $p_\psi = I_3\omega_{z'}^b$

$$\omega_{z'}^b \geq \frac{2}{I_3}\sqrt{MglI_1\cos\theta_o}$$

we can have steady precession at a fixed angle only if spin $\left(\omega_{z'}^b\right)$ is greater than the value given. For $\theta = \theta_o$, the equation for $\dot{\phi}$ becomes

$$\dot{\phi}_o = \frac{\beta}{I_1\sin^2\theta_o}$$

We have two possible values for $\dot{\phi}_o$ corresponding to the two roots of β

$$\begin{aligned}\dot{\phi}_o(+) &\rightarrow \text{fast precession} \\ \dot{\phi}(-) &\rightarrow \text{slow precession}\end{aligned}$$

in the limit where $\omega^b_{z'}$ is large (a fast spinning top) the second term in the radical in the expression for β is small, and we can expand and get (after some manipulation)

$$\dot{\phi}_o\,(+) \approx \frac{I_2\omega^b_z}{I_1\cos\theta_o}$$

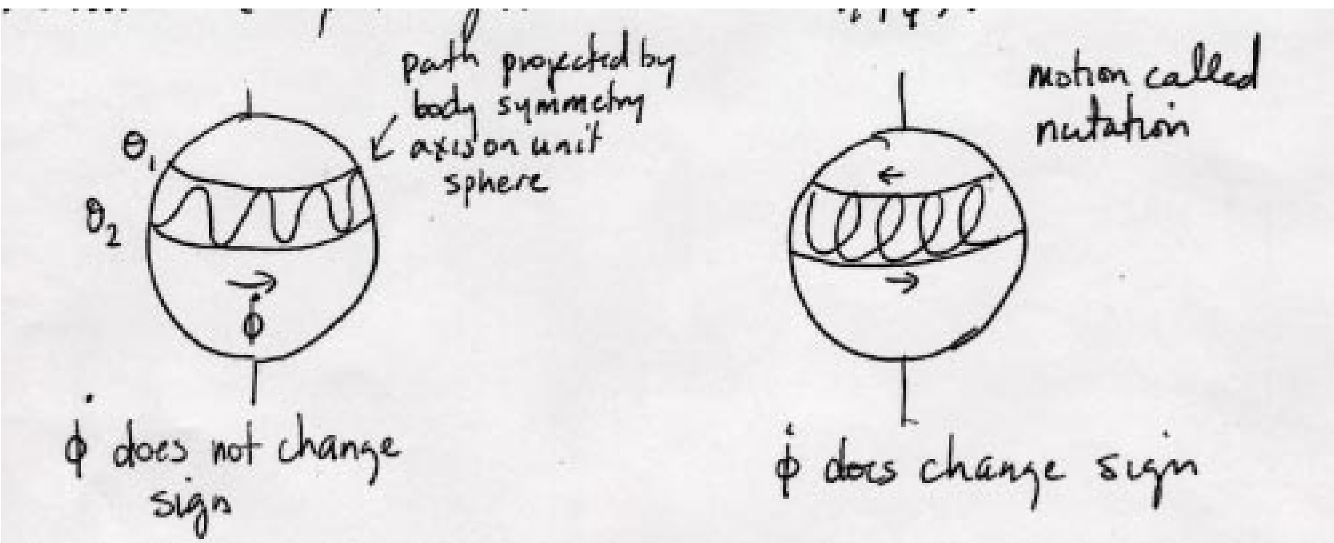
$$\dot{\phi}_o\,(-) \approx \frac{Mgh}{I_3\omega^b_{z'}}$$

Usually we observe the slower of the two precessional frequencies. The above applies for $\theta_o < \frac{\pi}{2}$. If $\theta_o > \frac{\pi}{2}$, the $\sqrt{}$ is always positive so there is no limiting condition on $\omega^b_{z'}$. $\sqrt{}$ is > 1 in this case, and $\dot{\phi}_o$ for fast and slow precession have opposite signs. For $\theta_o > \frac{\pi}{2}$ fast precession is in the same sense as for $\theta_o < \frac{\pi}{2}$ but slow precession has opposite sense.

For the general case where $\theta_1 < \theta < \theta_2$

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

This implies $\dot{\phi}$ may or may not change sign as θ varies between its limits (depending on the ratio p_ϕ/p_ψ).



If $\dot{\phi}$ does change sign, the angular velocity of precession has opposite signs at θ_1, θ_2 so get a looping motion.

In the special case where

$$(p_\phi - p_\psi\cos\theta)\,|_{\theta=\theta_1} = 0$$

$\dot{\phi}|_{\theta_1} = 0, \dot{\theta}|_{\theta_1} = 0$ and we get a cusp-like motion. This corresponds to the usual case starting a top (drop it with $\dot{\phi} = 0$). Other cases correspond to having an initial $\dot{\phi}$.