1 Topic 1: Hamiltonian Dynamics

Reading: Hand & Finch Chapter 5

Hamiltonian approach to mechanics is based on replacing $\dot{q}$ with p in the fundamental equations. The basic reason for the elimination of $\dot{q}$ in favor of p is that the velocity is a kinematic variable, whereas the momentum is a dynamical variable. Being hit by a grain of sand going 20 miles/hr is a lot different than being hit by a truck that is going 20 miles/hr. Let's look at how we do this replacement of variables for a simple case.

From 1-dimensional mechanics, we have (in cartesian co-ordinates)

$$\dot{p} = F = -\frac{\partial V}{\partial x} \quad \text{(for conservative forces)} \tag{1}$$

where

$$p = m\dot{x} \Longrightarrow \dot{x} = \frac{p}{m} \tag{2}$$

Note that the kinetic energy is $T = \frac{1}{2}m\dot{x}^2$ which can be expressed in terms of p only.

$$T = \frac{p^2}{2m} \tag{3}$$

and so

$$\dot{x} = \frac{\partial T}{\partial p} \tag{4}$$

Now lets look at several dimensions under elementary conditions (ie. time-independent L, conservative forces etc etc.). Let's compare the Lagrangian to the Hamiltonian, where we think of $L(q_i, \dot{q}_i; t)$, $H(p_i, q_i, t)$. We can deduce for simple situations what the equations of motion will be for the Hamiltonian case (we of course know already what they are for the Lagrangian.

Lagrange: $L = T - V = L(q_i, \dot{q}_i; t)$	Hamilton: $H = T + V = H(p_i, q_i, t)$
EOM: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$	EOM:
$p_i = \frac{\partial L}{\partial \dot{q}_i}$	$\dot{p}_i = -\frac{\partial H}{\partial q_i}$ $\left[-\frac{\partial V}{\partial x}\right]$
$\dot{p}_i = \frac{\partial L}{\partial q_i}$	$\dot{q}_i = +\frac{\partial H}{\partial p_i}$ $\left[\frac{\partial T}{\partial p}\right]$

1.1 Deriving Hamilton's Equations

The above is what you really need to remember. Let's look at a more general way of deriving the above. First, let's motivate the approach by looking at an analogous example from thermodynamics. If you recall in thermo we're always changing variables depending on whether we have a situation that is at constant pressure, or constant volume, etc.. If you express the internal energy of a simple system (a gas) in terms of the entropy and volume, for a reversible process you have

$$dU = TdS - pdV \tag{5}$$

where $U = U(S,V)$. So $T = \frac{\partial U}{\partial S}|_V,\ p = -\frac{\partial U}{\partial V}|_S$.

For experiments performed at constant volume, the heat ($dQ = TdS$) added just goes into a change in the internal energy. However, many terrestial experiments are performed at constant *pressure*. We therefore need to transform our function $U(S,V)$ into some other function $H(S,p)$ (=enthalpy, unfortunately denoted by the same letter as the Hamiltonian), similar to the way we transformed to equations involving the Hamiltonian. We want a thermodynamic variable of the form:

$$dH = TdS + \_\_\_dp \tag{6}$$

which is a lot more convenient, since at constant p, the enthalpy (or heat function) is what increases by the amount of heat added.

The trick is to write

$$dU = TdS - d(pV) + Vdp \tag{7}$$

or

$$dH = TdS + Vdp \tag{8}$$

where

$$H = H(S,p) = U(S, V(p,S)) + pV \tag{9}$$

We get $V(p,S)$ by solving the equation $p = -\frac{\partial U(S,V)}{\partial V}$ for V in terms of p, S.

In the same way we can eliminate velocity in mechanics in favor of momentum and introduce a new function, the Hamiltonian, a function of $H(p_i, q_i, t)$. Here's how it goes. We start with the Lagrangian

$$L = L(q_i, \dot{q}_i, t) \tag{10}$$

Define

$$p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \tag{11}$$

Keep in mind that although we by custom donote this with the same symbol as regular momentum, it's really the generalized momentum, and can be quite different. The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{12}$$

Then

$$\begin{aligned} dL &= \frac{\partial L}{\partial q_i}|_{\dot{q}_j,t}\, dq_i + \frac{\partial L}{\partial \dot{q}_i}|_{q_j,t}\, d\dot{q}_i + \frac{\partial L}{\partial t}|_{\dot{q}_j,q_j} \\ &= \frac{\partial L}{\partial q_i}|_{\dot{q}_j,t}\, dq_i + d(p_i\dot{q}_i) - \dot{q}_i dp_i + \frac{\partial L}{\partial t}|_{\dot{q}_j,q_j} \end{aligned}$$

or define the new function $H = p_i\dot{q}_i - L$. To make the dependences explicit:

$$H(p_i, q_i, t) = p_i\dot{q}_i(p_j, q_j, t) - L(q_i, \dot{q}_i(p_j, q_j, t), t) \tag{13}$$

then

$$
\begin{aligned}
dH &= -\frac{\partial L}{\partial q_i}|_{q_j,t} dq_i + \dot{q}_i dp_i - \frac{\partial L}{\partial t} dt \\
&= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt
\end{aligned}
$$

by definition we have

$$
dH = \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial t} dt \tag{14}
$$

so we can equate terms:

$$
\frac{\partial H}{\partial p_i} = \dot{q}_i; \quad -\frac{\partial H}{\partial q_i} = \dot{p}_i; \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{15}
$$

These are Hamilton's equations.

Finally we can use the general differential to write

$$
\begin{aligned}
\frac{dH}{dt} &= \frac{\partial H}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial H}{\partial t} \\
&= -\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial H}{\partial t} \\
&= 0 + \frac{\partial H}{\partial t}
\end{aligned}
$$

so,

$$
\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{16}
$$

and we have a familiar result: H is conserved if L has no exlicit time dependence.

The above two examples use Legendre transformations to change variables - a common mathematical trick.

We can take another approach to deriving the same equations which doesn't use the Legendre transformation trick. From

$$
p_i = \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \tag{17}
$$

we solve for

$$
\dot{q}_i = \Phi_i (p_i, q_i, t) \tag{18}
$$

where we used Φ to emphasize the functional form rather than the physical significance. (Note, I've assumed it is possible to solve for $\dot{q}_i$.) Then write $L = L(q_i, \Phi(p_i, q_i, t), t)$ so

$$
\begin{aligned}
\frac{\partial L}{\partial q_k}|_p &= \frac{\partial L}{\partial q_k}|_\Phi + \frac{\partial L}{\partial \Phi_i}|_q \frac{\partial \Phi_i}{\partial q_k}|_p \\
&= \frac{d}{dt}(p_k) + p_i \frac{\partial \dot{q}_i}{\partial q_k}|_p \\
&= \dot{p}_k + \frac{\partial (p_i \dot{q}_i)_p}{\partial q_k}
\end{aligned}
$$

or

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}|_p \quad , \; H = p_i \dot{q}_i - L \tag{19}$$

Next

$$\begin{aligned} \frac{\partial L}{\partial p_i}|_q &= \frac{\partial L}{\partial \Phi_k}|_q \frac{\partial \Phi_k}{\partial p_i}|_q \\ &= p_k \frac{\partial \dot{q}_k}{\partial p_i}|_q \\ &= \frac{\partial \left(p_k \dot{q}_k\right)}{\partial p_i}|_q - \dot{q}_i \end{aligned}$$

or

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad H = p_i \dot{q}_i - L \tag{20}$$

1.2 Why are Hamilton's Equations Useful?

What have we accomplished by transforming our representation from $q, \dot{q} \rightarrow p, q$? Just as Lagrangian mechanics is useful in practice because of the freedome it provides in the coordinate used to kinematically define a system, so Hamiltonian mechanics allow a yet larger latitude of transformations $(p_i, q_i) \rightarrow (p_i', q_i')$ which can be advantageously used. However, the freedom of $q_i \rightarrow q_i'$ with no change in the equations of motion for any reasonable transformation is is lost in Hamiltonian mechanics.

Only a subset of all possible transformations

$$(p_i, q_i) \rightarrow (p_i', q_i') \tag{21}$$

give rise to Hamilton's equations in the primed variables. Of course we can define *any* smooth transformation

$$\begin{aligned} P_i &= P_i \left(p_j, q_j, t\right) \\ Q_i &= Q_i \left(p_j, q_j, t\right) \\ T &= T \left(p_j, q_j, t\right) \end{aligned}$$

and get equations of motion

$$\begin{aligned} \frac{dP_i}{dT} &= f_i \left(P_j, Q_j, T\right) =? - \frac{\partial K}{\partial Q_i} \\ \frac{\partial Q_i}{\partial T} &= g_i \left(P_j, Q_j, T\right) =? \frac{\partial K}{\partial P_i} \end{aligned}$$

But does a Hamiltonian exist such that

$$\begin{aligned} \frac{dP_i}{dT} &= -\frac{\partial K}{\partial Q_i} \\ \frac{\partial Q_i}{\partial T} &= \frac{\partial K}{\partial P_i} \end{aligned}$$

?

From above, a necessary condition is

$$\frac{\partial f_i}{\partial P_j} = -\frac{\partial g_j}{\partial Q_i} \tag{22}$$

(This follows from $\frac{\partial^2 K}{\partial Q_i \partial P_j} = \frac{\partial^2 K}{\partial P_j \partial Q_i}$). This will not be satisfied in general. If it is, then the tranformation is said to be *canonical.* The Hamiltonian formulation of mechanics is in fact sometimes called canonical mechanics. If you are a mathematician, you may have seen the whole subject by another name. What physicists call a cononical transformation is a sympletic diffeomorphism to a mathematiciaion, intimately related to the symplectic group.

The most important use of Hamilton's formulation is that it gives insight into the formal mathematical structure of classical mechanics, and shows its relation to wave propagation (Hamilton's original interest), and so can be used as a stepping stone to quantum mechanics. Had there been an experimental rationale, Hamilton could have invented a form of quantum mechanics in the 1820's.

1.3 Some Examples

Lets do some examples to get the idea of how the application of Hamilton's equations goes in practice. If we have n degrees of freedom, then we have $2n$ first-order differential equations in Hamilton's formulation, which replace the n second order Lagrange equations. To use Hamilton's equations (also called the canonical equations due to the symmetry) , we must first construct the Hamiltonian as a function of the generalized coordinates and momenta. It may be possible in some instances to do this directly. In more complicated cases, it may be necessary first to set up the Larangian and then to calculate the generalized momenta according to $p_i = \frac{\partial L}{\partial \dot{q}_j}$.

1.3.1 Simple Harmonic Oscillator

For an SHO

$$L = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 \tag{23}$$

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, and

$$H = p\dot{x} - \frac{k}{2}x^2 = \frac{p^2}{2m} + \frac{k}{2}x^2 \tag{24}$$

Hamilton's equations for $\dot{q}$ and $\dot{p}$ are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \; \dot{p} = \frac{-\partial H}{\partial x} = -kx \tag{25}$$

If we differentiate the right hand equation and substitute, we get the familiar EOM: $m\ddot{x} + kx = 0$.

1.3.2 A particle constrained to move on a cylinder

Let's use the Hamiltonian method to find the equations of motion of a particle of mass m constrained to move on the surface of a cylinder defined by $x^2 + y^2 = R^2$. The particle is subject to a force directed toward the origin and proportional to the distance of theparticle from the origin: $\mathbf{F} = -k\mathbf{r}$.

We'll use cylindrical co-ordinates z, R, ϕ. The potential energy is

$$V = \frac{1}{2}kr^2 = \frac{1}{2}k\left(x^2 + y^2 + z^2\right) = \frac{1}{2}k\left(R^2 + z^2\right) \tag{26}$$

In cylindrical co-ordinates

$$v^2 = \dot{R}^2 + R^2\dot{\theta}^2 + (\frac{dz}{dt})^2 \tag{27}$$

R is a constant, so the kinetic energy is

$$T = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) \tag{28}$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right) \tag{29}$$

Our generalized co-ordinates are θ, z, and the generalized momenta are

$$\begin{aligned} p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \\ p_z &= \frac{\partial L}{\partial \dot{z}} = m\dot{z} \end{aligned}$$

Because the system is conservative and the equations of transformation between rectangular and cylindrical coordinates do not explicitly involve the time, the Hamiltonian H is just the total energy expressed in terms of the variables θ, p_θ, z, p_z:

$$\begin{aligned} H\left(z, p_\theta, p_z\right) &= T + V \\ &= \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}kz^2 \end{aligned}$$

where we have noticed that there is no dependence on θ, and we suppress the constant term $\frac{1}{2}kR^2$. The canonical equations of motion are therefore:

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \\ \dot{p}_z &= -\frac{\partial H}{\partial z} = -kz \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \\ \dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \end{aligned}$$

We see immediately that p_θ is a constant - conservation of angular momentum about the $z-$ (symmetry) axis. We can get the EOM for z from $\dot{p}_z = -kz = m\ddot{z}$, so

$$\ddot{z} + \omega^2 z = 0 \tag{30}$$

so the motion in the z direction is simple harmonic.

Of course these equations could have been found directly from the Lagrangian method. Often it is faster to do this, however in celestial mechancis, particularly in the event that the motions are subject to perturbations, it proves convenient to formaulte the problem in terms of Hamiltonian dynamics. Also, we see that in the Hamiltonian formulation if q_k is cyclic, p_k is constant, and we reduce the number of equations to be solved. We also readily identify constants of the motion. More on this later.

1.3.3 Charged Particle in an E-M field

This example shows how to construct the Hamiltonian. We'll show later that the Lagrangian for a charged particle in an E-M field is given by

$$L = \frac{1}{2}m\dot{x}_i\dot{x}_i - e\phi + \frac{e}{c}\dot{x}_i A_i \tag{31}$$

where A_i are the components of the vector potential, and ϕ is the electrostatic potential. We'll prove this in a moment, but first let's go through the exercise of getting the Hamiltonian. We get the canonical momentum from taking the partial derivatives of L -

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i \tag{32}$$

so we eliminate $\dot{x}$ from the Lagrangian by inverting the above:

$$\dot{x}_i = \frac{1}{m}\left(p_i - \frac{e}{c}A_i\right) \tag{33}$$

and get H from $H = p_i\dot{q}_i - L$

$$\begin{aligned} H &= p_i\frac{1}{m}\left(p_i - \frac{e}{c}A_i\right) - \frac{1}{2m}\left(p_i - \frac{e}{c}A_i\right)\left(p_i - \frac{e}{c}A_i\right) + e\phi - \frac{e}{cm}\left(p_i - \frac{e}{c}A_i\right)A_i \\ &= \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\phi = T + V \end{aligned}$$

Note: $\mathbf{p}$ is the cononical momentum, *not* $m\mathbf{v}$! Whether H is constant in time depends on whether the fields are static. Given $\mathbf{A}$, we could then apply Hamilton's equations to get the EOM for the particle.

Now let's take a short detour and demonstrate that the above is the correct L from the EOM. The EOM is

$$m\ddot{\mathbf{x}} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{34}$$

where

$$\begin{aligned}\mathbf{E} &= -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \boldsymbol{\nabla} \times \mathbf{A}\end{aligned}$$

The other two Maxwell equations, $\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$, $\nabla \cdot \mathbf{B} = 0$ provide equations for determining $\mathbf{A}, \phi$. So,

$$\begin{aligned}\frac{\mathbf{v}}{c} \times \mathbf{B} &= \frac{\mathbf{v}}{c} \times \boldsymbol{\nabla} \times \mathbf{A} \\ &= \boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) - \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\mathbf{A}\end{aligned}$$

and

$$\begin{aligned}\left(\frac{\mathbf{v}}{c} \times \mathbf{B}\right)_i &= \frac{\partial}{\partial x_i}\left(\frac{\dot{x}_j}{c}A_j\right) - \left(\frac{\dot{x}_j}{c}\frac{\partial}{\partial x_j}\right)A_i \\ &= \frac{\dot{x}_j}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\end{aligned}$$

Also,

$$E_i = -\frac{\partial \phi}{\partial x_i} - \frac{1}{c}\frac{\partial A_i}{\partial t} \tag{35}$$

So

$$m\ddot{x}_i = e\left(\frac{-\partial \phi}{\partial x_i} - \frac{\partial A_i}{c\,\partial t}\right) + e\frac{\dot{x}_j}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right) \tag{36}$$

The Lagrangian

$$L = \frac{1}{2}m\dot{x}_i\dot{x}_i - e\phi + \frac{e}{c}\dot{x}_i A_i \tag{37}$$

Check this:

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = \frac{d}{dt}\left(m\dot{x}_i + \frac{e}{c}A_i\right) + e\frac{\partial \phi}{\partial x_i} - \frac{e}{c}\left(\dot{x}_j\frac{\partial A_j}{\partial x_i}\right) \tag{38}$$

or

$$\begin{aligned}m\ddot{x}_i &= \frac{-e}{c}\frac{\partial A_i}{\partial x_j}\dot{x}_j - \frac{e}{c}\frac{\partial A_i}{\partial t} - e\frac{\partial \phi}{\partial x_i} + \frac{e}{c}\dot{x}_j\frac{\partial A_j}{\partial x_i} \\ &= e\left(\frac{-\partial \phi}{\partial x_i} - \frac{\partial A_i}{c\partial t}\right) + \frac{e}{c}\dot{x}_j\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\end{aligned}$$

QED.

Hamilton's formulation is particularly useful for considering motion in a non-inertial frame. Read the example of the bug on the turntable in your book. In the case of a charged particle in a uniform B-field, we know it undergoes helical motion - spiraling around the B-field lines. Let's analyze the motion:

For a uniform B-field:

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{B} \times \mathbf{r}\right) \tag{39}$$

for a uniform field in the z-direction,

$$A_x = -\frac{B}{2}y, \;\; A_y = \frac{B}{2}x, \quad A_z = 0 \tag{40}$$

Stick this into the Hamiltonian:

$$\begin{aligned} H &= \frac{p^2}{2m} - \frac{e}{2mc^2}\left(xp_y - yp_x\right)B + \frac{e^2}{2mc^2}\left(\frac{1}{4}r^2B^2\right) + e\phi \\ &= \frac{p^2}{2m} - \frac{e}{2mc^2}l_zB + \frac{e^2}{2mc^2}\left(\frac{1}{4}r^2B^2\right) + e\phi; \;\; l_z \equiv \left(xp_y - yp_x\right) \end{aligned}$$

If we look at this form for H with $\phi = 0$, we notice that it consists of one part describing simple harmonic motion in 3-D we just saw above plus $-\frac{e}{2mc^2}l_zB$. The latter part is associated with the circular motion.

In your book, it is shown that for motion in a rotating frame the H transforms as

$$H_{\omega\neq 0} = H_{\omega=0} - \omega I_z \tag{41}$$

If we define ω_L,

$$\omega_L = -\frac{eB}{2mc} \tag{42}$$

(the Larmor frequency), then the motion is described by

$$H_{\omega\neq 0} = \frac{p^2}{2m} + \frac{e^2r^2B^2}{8mc^2} - \omega_L l_z \tag{43}$$

and if we transform into a frame rotating at ω_L,

$$H_{rotating} = \frac{p^2}{2m} + \frac{e^2r^2B^2}{8mc^2} \tag{44}$$

1.4 Cyclic Coordinates and Conservation Theorems

Last term we considered the relationship between cyclic coordinates and conservation laws in the context of Lagrangian mechanics. You will recall that a cyclic coordinate is one that doesn't appear explicitly in the Lagrangian. The conjugate momentum to this cyclic coordinate is constant (this follows from the Lagrange equations). What can we say about H and cyclic coordinates?

$$\dot{p}_j = \frac{\partial L}{\partial q_j} = -\frac{\partial H}{\partial q_j} \tag{45}$$

A coordinate that is cyclic will therefore also be absent from the Hamiltonian. The momentum conservation theorems we derived from L therefore also apply to the Hamiltonian formulation with only a substitution of H for L. We saw this for the cyclic angular coordinate ϕ and conservation of angular momentum in the particle moving on a cylinder example above.

There is, however, a difference in the two formulations when we consider the time-dependence of L and H and how they behave under coordinate transformations. We have already seen that if L is not an explicit function of t then H is a constant of the motion – seen directly from the relationship we derived last lecture:

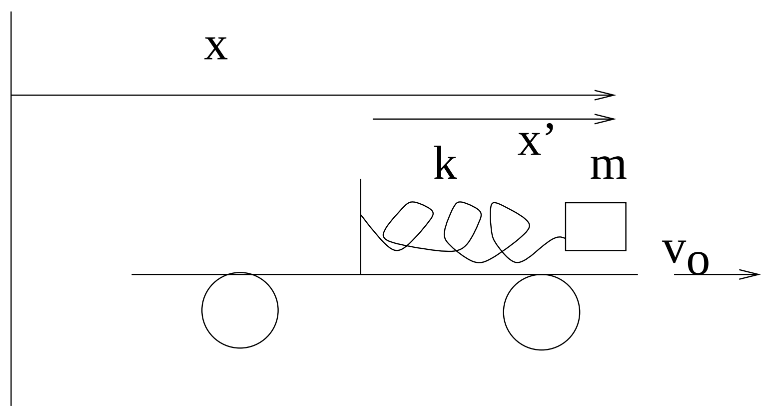
$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \tag{46}$$

We also saw last term that if the the equations of transformation that define the generalized coordinates

$$\mathbf{r}_m = \mathbf{r}_m \left(q_1 ..., q_n; t\right) \tag{47}$$

don't depend explicitly on time (in the above r_m are the components of the cartesian system), and if the potential is velocity-independent then H is the total energy, $H = T + V$. Note that H being constant, and H being equal to the total energy result from different sets of conditions. We can have a constant H not equal to the total energy, or a time-varying H equal to the toal energy.

You should also note that a change of generalized coordinates can change the functional appearance of L, but cannot change its actual value. On the other hand, a change of coordinates can lead not only to a different functional form for H, but can also change its value. For one set of generalized coordinates H may be conserved, but for another it could vary in time.



To illustrate these points, consider the 1-D system of a mass on a spring moving at constant velocity v_o along the x-axis. Let x be the coordinate of the mass in the fixed frame, and x' be the coordinate in the moving system:

We have

$$L\left(x, \dot{x}, t\right) = T - V = \frac{m\dot{x}^2}{2} - \frac{k}{2}\left(x - v_o t\right)^2 \tag{48}$$

The corresponding EOM we can get from the Lagrange equation:

$$m\ddot{x} = -k\left(x - v_o t\right) \tag{49}$$

If we change to the moving system by transforming coordinates

$$x' = x - v_o t \tag{50}$$

the EOM becomes

$$m\ddot{x}' = -kx' \tag{51}$$

and an observer moving with the spring sees simple harmonic motion.

Now lets look at the Hamiltonian formulation. x is the cartesian coordinate and the pootential does not involve the generalized velocities, therefore H in the fixed system is the sum of kinetic and potential energies:

$$H = T + V = \frac{p^2}{2m} + \frac{k}{2}(x - v_o t)^2 \tag{52}$$

and although H is the total energy, it depends on time (why?).

If L is formulated from the start in the x' system,

$$L(x', \dot{x}') = \frac{1}{2}m\dot{x}'^2 + m\dot{x}'v_o + \frac{1}{2}mv_o^2 - \frac{1}{2}kx'^2 \tag{53}$$

and $p' = m(\dot{x}' + v_o)$. Substituting $\dot{x}' = \frac{1}{m}p' - v_o$ into L

$$\begin{aligned} H &= p'\left(\frac{1}{m}p' - v_o\right) - \frac{m}{2}\left(\frac{1}{m}p' - v_o\right)^2 - mv_o(\frac{1}{m}p' - v_o) - \frac{1}{2}mv_o^2 + \frac{1}{2}kx'^2 \\ &= \frac{(p' - mv_o)^2}{2m} + \frac{kx'^2}{2} - \frac{mv_o^2}{2} \end{aligned}$$

Now H' is not the total energy of the system, but it is conserved. Except for the last term, it is the total energy of motion of the particle relative to the fixed point of the spring. The Hamiltonians are different in magnitude and time behavior, but lead to the same EOM.

1.5 Noether's Theorem

There is a theorem originally derived by Emmy Noether formalizing the relationship between symmetry properties and conserved quantities. We can think of a symmetry as some continuous transformation of the system which leaves the Lagrangian unchanged. We describe the transformation by parametrizing the coordinate transformation as follows: if $q(t)$ is the solution to the problem for a particular Lagrangian, then we can describe the coordinate transformation using a parameter s that can be continuously varied, so that the transformed coordinate is

$$Q(s, t) \tag{54}$$

where $q(t) = Q(0, t)$ by definition. The new Lagrangian corresponding to the transformed system is $L' = L'\left(Q(s,t), \dot{Q}(s,t)\right)$, and if the Lagrangian is invariant, then

$$L' = L\left(Q(s,t), \dot{Q}(s,t)\right) = L(q, \dot{q}) \tag{55}$$

(where we ignore any explicit time dependence for simplicity). Now L must not depend on s:

$$\frac{d}{ds}L\left(Q(s,t),\dot{Q}(s,t)\right)=0 \tag{56}$$

By the chain rule:

$$\begin{aligned}\frac{dL}{ds} &= \frac{\partial L}{\partial Q}\frac{\partial Q}{\partial s}+\frac{\partial L}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial s}\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right)\frac{\partial Q}{\partial s}+\frac{\partial L}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial s}\\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\frac{\partial Q}{\partial s}\right)=0\end{aligned}$$

So if we evaluate the quantity in parentheses at $s=0$ for convenience,

$$p\frac{\partial Q}{\partial s}|_{s=0}=const \tag{57}$$

We can generalize this to an arbitrary number of dimensions:

$$I_j\equiv\sum_{k=1}^{N}p_k\frac{dQ_k}{ds_j}|_{s_l=0}=const \tag{58}$$

Each of these I_j's is a constant of the motion. For example, if the system is rotationally invariant, then the three I_j's would be the three components of the angular momentum (and the s's correspond to three angles).

The importance of Noether's theorem is that it gives us a prescription for finding conserved quantities and their associated symmetries where they may otherwise be difficult to recognize. Note that we use $p_k=\frac{\partial L}{\partial \dot{q}_k}$ as we evaluate the partials at $s=0$. This removes the requirement that we be able to find the associated transformation for the conjugate momenta.

1.6 Phase Space and Liouville's Theorem

We discussed last term how the generalized coordinates can be used to define a $2N$ dimensional *phase space* (sometimes referred to as Hamiltonian phase space) where the coordinates of the axes are the $q_i,\dot{q}_i$, and each point on the phase space diagram represents a certain state of the system. Similarly we can define the space using the q_i,p_i. If, at a given time, the position and momenta of all the particles in a system are known, then with these as initial conditions, the subsequent motion of the system is completely determine – ie. the representative point moves along a unique phase path.

For large and/or complex systems, it is often practically impossible to determine theinitial conditions for each particle. We therefore must devise an alternative approach to study the dynamics of such systems: statistical mechanics. The Hamiltonian formulation is ideal for the statistical study of complex systems. Liouville's theorem is an example of this.

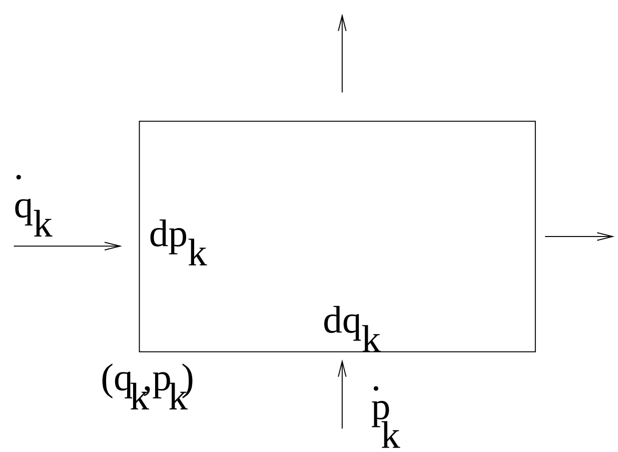
For a large collection of particles (e.g. gas molecules) we are unable to identify a specific point in phase space corresponding to the state of the system. We can however fill the phase space with a points representing the possible states of the system (much of the space may be excluded due to total energy conservation, etc). We substitute the discussion of the *trajectory of a single system* by the discussion of an *ensemble of equivalent systems*. Each representative point corresponds to a single system of the ensemble, and the motion of a particular point represents the independent motion of that system. No two of these paths in phase space can cross, since if they did a given system could evolve in more than one direction, violating the deterministic nature of classical mechanical systems.

We may consider the points to be sufficiently numerous that we can define a density in phase space, ρ. The volume elements used to define ρ must be sufficiently large so that the ρ varies continuously. The number S of systems whose representative points lie in a volume dv is

$$S = \rho\ dv \tag{59}$$

where

$$dv = dq_1 dq_2 ... dq_N\ dp_1 dp_2 .... dp_N \tag{60}$$



Consider an element of area in the $q_k - p_k$ plane in phase space. The number of points moving across the left-hand edge into the area per unit time is

$$\rho \frac{dq_k}{dt} dp_k = \rho \dot{q}_k dp_k \tag{61}$$

and the number moving across the lower edge is

$$\rho \frac{dp_k}{dt} dq_k = \rho \dot{p}_k dq_k \tag{62}$$

so the total number of points moving into the area is:

$$\rho \left(\dot{q}_k dp_k + \dot{p}_k dq_k \right) \tag{63}$$

If we take the element to be small enough that we can apply a Taylor series expansion, then the total number of points flowing out of the area is

$$\left[\rho\dot{q}_k + \frac{\partial}{\partial q_k}\left(\rho\dot{q}_k\right)dq_k\right]dp_k + \left[\rho\dot{p}_k + \frac{\partial}{\partial p_k}\left(\rho\dot{p}_k\right)dp_k\right]dq_k \tag{64}$$

so the total increas in density in the element $dq_k dp_k$ per unit time is

$$\frac{\partial\rho}{\partial t}dq_k dp_k = -\left[\frac{\partial}{\partial q_k}\left(\rho\dot{q}_k\right) + \frac{\partial}{\partial p_k}\left(\rho\dot{p}_k\right)\right]dq_k dp_k \tag{65}$$

summing over all k,

$$\frac{\partial\rho}{\partial t} + \sum_{k=1}^{N}\left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \rho\frac{\partial\dot{q}_k}{\partial q_k} + \frac{\partial\rho}{\partial p_k}\dot{p}_k + \rho\frac{\partial\dot{p}_k}{\partial p_k}\right) = 0 \tag{66}$$

From Hamilton's equations:

$$\frac{\partial\dot{q}_k}{\partial q_k} = \frac{-\partial\dot{p}_k}{\partial p_k} \tag{67}$$

which we substitute in above

$$\frac{\partial\rho}{\partial t} + \sum_{k=1}^{N}\left(\frac{\partial\rho}{\partial q_k}\dot{q}_k + \frac{\partial\rho}{\partial p_k}\dot{p}_k\right) = 0 \tag{68}$$

By definition this is just the total time derivative of ρ, so we have

$$\frac{d\rho}{dt} = 0 \tag{69}$$

This is called Liouville's theorem, which states that the density of points in phase space corresponding to the motion of a system of particles remains constant during the motion. Note that we used Hamilton's equation to derive this result, and it holds only in phase space, so we must use Hamiltonian dynamics (rather than Lagrangian dynamics) to discuss ensembles in statistical mechanics.

Liouville's theorem is useful in gaseous systems, charged-particle accelerators, and also in stellar dynamics.