

# 1 Topic 2: Solving Poisson's Equation, Intro to Green's Functions, Image Charge

Reading Assignment: Jackson Chapter 2.1 - 2.8

## 1.1 Boundary Conditions and Uniqueness of Solutions

Laplace's equation does not by itself determine  $\Phi$ , we need to supply a suitable set of boundary conditions. This raises the question: what are appropriate boundary conditions, sufficient to determine the answer but not so strong as to create inconsistencies. Should we specify the value of the function at the boundary, the value of the function as well as its derivative, the value of the function at some boundaries and the derivative at others? With a 2 or 3-D PDE it's not so easy to see what is acceptable. If we go back to our drum-head example, is the shape of the rubber membrane uniquely determined by the frame we stretch it over, or can it snap back and forth from one stable configuration to another (like the lid of a canning jar)? As your intuition would suggest,  $\Phi$  is uniquely determined by its value at the boundary – but other boundary conditions can also be used. The proof that a proposed set of boundary conditions suffices is given by the *uniqueness theorem*.

We can prove that the solution to Laplace's equation in some region is uniquely determined if  $\Phi$  is specified on all boundaries of the region.

Proof: Suppose there are two solutions to the Laplace equation:

$$\begin{aligned}\nabla^2\Phi_1 &= 0 \\ \nabla^2\Phi_2 &= 0\end{aligned}$$

both of which have the same (given) value on the surface. Then  $\Phi_3 = \Phi_1 - \Phi_2$  obeys the Laplace equation

$$\nabla^2\Phi_3 = \nabla^2\Phi_1 - \nabla^2\Phi_2 = 0 \tag{1}$$

We also know  $\Phi_3$  is zero on all boundaries (since  $\Phi_{1,2}$  are equal there). We saw that the Laplace equation permits no local maxima or minima, these must occur on the boundaries. So, the maximum and minimum of  $\Phi_3$  are both zero, and  $\Phi_3$  is zero everywhere, and hence

$$\Phi_1 = \Phi_2 \tag{2}$$

and we have one unique solution specified by the value of  $\Phi$  at the boundary.

What the uniqueness theorem says is that it doesn't matter by what means we come up with our solution, no matter how sleazy - if it satisfies the boundary condition it is the right answer.

We can also show that if we have a configuration of  $n$  conductors with an outer boundary, and we put some charge  $q_n$  on each, then the electric field is uniquely determined. What we want to prove can be stated as follows: In a region containing conductors and filled with a specified charge density  $\rho$ , the electric field is uniquely determined if the total charge on each conductor is given.

Proof: Suppose there are two fields satisfying the problem. Both obey Gauss's law

$$\begin{aligned}\nabla \cdot \mathbf{E}_1 &= \frac{1}{\epsilon_o} \rho \\ \nabla \cdot \mathbf{E}_2 &= \frac{1}{\epsilon_o} \rho\end{aligned}$$

If we construct a surface around each conductor,

$$\begin{aligned}\oint_{ith\ surface} \mathbf{E}_1 \cdot \mathbf{n} \, da &= \frac{1}{\epsilon_o} q_i \\ \oint_{ith\ surface} \mathbf{E}_2 \cdot \mathbf{n} \, da &= \frac{1}{\epsilon_o} q_i\end{aligned}$$

Likewise for the outer boundary (enclosing the charge distribution and all the conductors)

$$\begin{aligned}\oint_{boundary} \mathbf{E}_1 \cdot \mathbf{n} \, da &= \frac{1}{\epsilon_o} q_{tot} \\ \oint_{boundary} \mathbf{E}_2 \cdot \mathbf{n} \, da &= \frac{1}{\epsilon_o} q_{tot}\end{aligned}$$

looking at the difference

$$\mathbf{E}_3 = \mathbf{E}_1 - \mathbf{E}_2 \quad (3)$$

which obeys

$$\nabla \cdot \mathbf{E}_3 = 0 \quad (4)$$

between the conductors, and

$$\oint \mathbf{E}_3 \cdot \mathbf{n} \, da = 0 \quad (5)$$

over each boundary surface.

Although we don't know how the charge is distributed on each conductor, we do know each is an equipotential surface.  $\Phi_3$  is therefore constant over each conducting surface (but not necessarily the same constant). Note that  $\Phi_3$  need not be zero, since  $\Phi_{1,2}$  can be different - we just require the total charge to be the same. We know

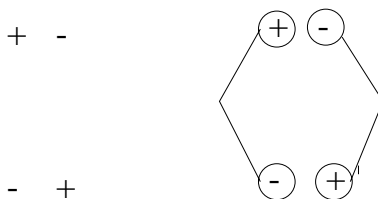
$$\nabla \cdot (\Phi_3 \mathbf{E}_3) = \Phi_3 (\nabla \cdot \mathbf{E}_3) + \mathbf{E}_3 \cdot (\nabla \Phi_3) = -E_3^2 \quad (6)$$

Integrate this over the region between conductors and apply the divergence theorem to the left side:

$$\int_V \nabla \cdot (\Phi_3 \mathbf{E}_3) \, d^3x = \oint_S \Phi_3 \mathbf{E}_3 \cdot \mathbf{n} \, da = - \int_V E_3^2 \, d^3x \quad (7)$$

The surface integral covers all boundaries of the region, including the conductors and the outer boundary.  $\Phi_3$  is constant over each surface, so bring it outside the integral, so

$$\int_V E_3^2 \, d^3x = 0 \quad (8)$$



since the integrand is never negative,  $E_3 = 0$ .

To illustrate our point, consider the simple, seemingly stable charge distribution

If we connect upper and lower +/- with a conducting wire, with this be stable? All we have to do to answer this is look at two neutral conducting wires - that is a solution to the problem, and must therefore be unique. So, when we connect them with wires the +/- charges must flow together and cancel.

We can also show that the field is uniquely determined when  $\rho$  is given, and either  $\Phi$  or the normal derivative  $\frac{\partial\Phi}{\partial n}$  is specified at each boundary. The boundaries need not be conductors, and  $\Phi$  need not be constant over the surface.

## 1.2 Green's Theorem and Green's Functions

We will proceed with what (for now) is a purely mathematical exercise. We want to convert the Poisson equation - a differential equation, into an integral equation, where we specify some charge distribution within some volume, as well as some boundary conditions on the surface of this volume.

If we have any two functions,  $\phi, \psi$ , we can show that

$$\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3x = \oint_S \phi \frac{\partial \psi}{\partial n} da \quad (9)$$

This is a purely mathematical derivation involving vector identities and the divergence theorem (see Jackson p36), called *Green's first identity*. If we interchange  $\phi$  and  $\psi$  and subtract, we get rid of the  $\nabla \phi \cdot \nabla \psi$ , and we have *Green's second identity*,

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3x = \oint_S \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] da \quad (10)$$

Now our goal is to turn the Poisson equation into an integral expression. We know the Poisson equation relates  $\nabla^2 \Phi$  to the charge density distribution. Given  $\rho$  in the volume,  $\Phi$  on the surface, we want to find  $\Phi$  in the interior. So, we want to replace one of our functions with  $\Phi$ . The question is what should the other one be? We would like to isolate  $\Phi(x)$ , so that we have an integral equation to solve directly for it. If we remember that

$$\nabla^2 \left( \frac{1}{R} \right) = -4\pi \delta(\mathbf{x} - \mathbf{x}') \quad (11)$$

and use  $\frac{1}{R}$  as our other function, then we'll have the term

$$\int_V \Phi(\mathbf{x} - \mathbf{x}') (-4\pi \delta(\mathbf{x} - \mathbf{x}')) d^3x' = \Phi(\mathbf{x}) \quad (12)$$

and the term

$$\int_V \frac{1}{R} \nabla'^2 \Phi(\mathbf{x} - \mathbf{x}') d^3 x' = \frac{1}{\epsilon_o} \int_V \frac{1}{R} \rho(x') d^3 x' \quad (13)$$

on the left, related to surface integrals on the right, which we can rearrange to solve for  $\Phi(x)$  (where the point  $x$  lies in the volume  $V$ ):

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_o} \int_V \frac{\rho(\mathbf{x}')}{R} d^3 x' + \frac{1}{4\pi} \oint_S \left[ \frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left( \frac{1}{R} \right) \right] da' \quad (14)$$

Now we have looked at the uniqueness of the solution of the potential if we specify  $\Phi$  everywhere on the bounding surface – this is *all we need to specify* (in addition to  $\rho$  inside the volume) in order to uniquely determine  $\Phi$ . If we specify anything more, the problem will be overdetermined. This is called *Dirichlet boundary conditions*. OR, we can specify  $\frac{\partial \Phi}{\partial n'}$  everywhere on the boundary to uniquely determine  $\Phi(\mathbf{x})$  (called *Neumann boundary conditions*). In this class, we will be considering almost exclusively problems with *Dirichlet conditions*.

If we look at the expression above, we can see that we are relating  $\Phi(\mathbf{x})$  to a volume integral containing the charge, and two surface integrals - one where  $\Phi$  is specified, and one where  $\frac{\partial \Phi}{\partial n'}$  is specified. In general this isn't going to help us solve problem, since to determine the problem uniquely, we will either want to specify  $\Phi$ , OR  $\frac{\partial \Phi}{\partial n'}$  on the surface. We can't specify both independently and still get a solution (the problem will be overdetermined).

This makes us consider using something more general than just  $\frac{1}{R}$  for our second function. We would like to use  $\frac{1}{R}$  plus some other function with Laplacian of zero inside the volume (so it doesn't contribute to our volume integral), but that has the properties on the surface that allow us to cancel one of the surface integrals in our equation above. We use

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}') \quad (15)$$

where we require

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0 \quad (16)$$

So we rewrite the expression for  $\Phi$  as

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_o} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d^3 x' + \frac{1}{4\pi} \oint_S \left[ G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'} - \Phi(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} \right] da' \quad (17)$$

Since we have allowed ourselves freedom in how we define  $G(\mathbf{x}, \mathbf{x}')$ , we can construct it such that we make one of the surface integrals vanish. For Dirichlet boundary conditions, we choose  $F$  such that

$$G(\mathbf{x}, \mathbf{x}') = 0 \text{ for } \mathbf{x}' \text{ on } S \quad (18)$$

and we have to solve

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_o} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d^3 x' + \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}) \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'} da' \quad (19)$$

It's not obvious how we construct  $G$  to accomplish our goal – solving for  $\Phi$  by constructing some function that satisfies the conditions we have established on the surface of our bounding volume. As we will see it is something of an art.

One thing that leads us to a useful method for some circumstances is noticing that  $F(\mathbf{x}, \mathbf{x}')$  satisfies the Laplace equation inside  $V$ , so represents a configuration with no charge inside the volume of interest, but that has boundary conditions with the appropriate properties on the surface (to cancel the parts of the surface integral we want to get rid of). Thus, it physically represents a charge distribution external to the volume we have defined, arranged so as to have the appropriate value on the surface.

### 1.3 The Method of Images

The method of image charges is a physically-intuitive way of "solving" for the Green's function. Here we consider our bounding surface to be conducting, so that it is at a constant (generally specified) potential. We divide space into an "interior" and "exterior" volume, where we are interested in the potential in one *or* the other. In the region of interest we place a charge distribution. Whatever this charge distribution, we know that the surface of the conductor must be an equipotential, and the field must be normal to the conducting surface. Charge will therefore move around on the conductor (be induced) so as to maintain the boundary conditions on the surface.

We ask, "what equivalent charge can we place somewhere outside the volume of interest that will have the effect of producing (in conjunction with our "real" charge) the specified boundary conditions? We then know from uniqueness that in the volume of interest the potential is given by the real plus the "image" charge configuration.

#### 1.3.1 Example: Charge above an infinite conducting plane (Jackson 2.1)

Class problems this week:

Jackson 2.1, 2.2, 1.15, 2.15