1 Topic 3: Non-integrable Systems and Chaos

Reading assignment: Hand and Finch Chapter 11

We have come to the final topic in classical mechanics: non-integrable systems and chaos. Of everything we have discussed, this is the most relevant to current research. It is a topic which, however, doesn't lend itself to analytical problem solving, and we will be discussing many characteristics of nonlinear systems that we will not "prove", but rather you will be developing a feeling for the general behavior. A big part of this will be the homework, where you will study a particular map or system in detail, possibly using the computer. I am assigning the homework early, and you should begin it as soon as possible.

1.1 Introduction

So far, the theme of this course has been: the main problem of classical mechanics is to "solve" for the phase space trajectory of the system – i.e. reduce it to *quadrature* (the evaluation of 1-D integrals that can be easily be done numerically).

Now we come to the great scandal – this is almost never possible. The mechanics problems that can be solved analytically on which we have spent so much time (Kepler, gyroscopes, coupled oscillators in the linear regime) are in a sense a set of measure zero – *very* exceptional. If the number of d.o.f is $N \geq 2$, a generic small perturbation

$$H = H_o + \varepsilon H_1 \tag{1}$$

where the system associated with H_o is solvable destroys the integrability of the system.

Furthermore, the non-integrable systems share a common, crucial feature: (exponentially) *sensitive dependence on initial conditions.* If we consider evolving a bundle of trajectories in phase space, for an integrable system the bundle spreads linearly with time.

But, for a nonintegrable system the spreading is exponential: $e^{t/T} = 10^{(\log e)t/T}$. This means that to evolve the system forward in time for time λT to one place accuracy, we need $\lambda (\log_{10} e)$ accuracy in the initial data. Our inability to specify initial data to arbitrary accuracy means that long-term behavior of the system is completely unpredictable – we say it is *chaotic.*

So, what chaos challenges is not just the quaint notion that physics problems should be exactly solvable, but the whole tradition of determinism in physics – the clockwork universe.

The traditional view is: give me the initial data and the Hamiltonian and I can tell you the state of the Universe at all later (or earlier) times. Chaos means this is not so. To propagate the system forward for longer and longer times requires more and more accurate knowledge of the initial state. Since we cannot even in principle know the initial state to infinite precision (a real number with an infinite number of digits – where would we write it down?), we cannot even in principle predict the behavior of the system arbitrarily far into the future.

Once we know to look, we can see chaos all around us – it is what makes the world beautiful and fascinating:

the dancing of the flames in the fireplace

the waves breaking on the shore

the clouds drifting by

These are situations that develop over and over from similar conditions, but because of sensitive dependence on initial conditions, we never see the same pattern repeat itself. That is why these things are fun to watch.

In one respect, the discovery that chaos is generic seems welcome, as it provides a basis for the assumption of ergodicity in classical statistical mechanics. Integrable systems are not ergodic – for a system with N dof there are N single-valued constants of the motion, and a generic trajectory fills only a N-dimensional torus in the $2N-$dimensional phase space. But the integrable systems are highly exceptional. We want to ask what is the typical behavior of a trajectory of a Hamiltonian system?

– Landau believed that N-tori are generic. In fact, he thought that all systems should be integrable and analytically solvable, a misunderstanding that mars the first section of the otherwise beautiful mechanics text by Landau and Lifshitz.

– At the other extreme, Fermi believed that all non-integrable systems are ergodic – a generic perturbation completely destroys the invariant tori.

The remarkably subtle and complex truth is somewhere inbetween. There are trajectories of a perturbed integrable system that fill a region of dimension $2N$, rather than an N torus, while there are also trajectories that remain confined on $N - tori$. When the perturbation is very weak ($\varepsilon << 1$), N-tori are "typical" but more and more "chaotic" trajectories appear as ε increases.

Some history:

1680's: Newton, trying to understand the motion of the earth-moon-sun system considers the problem of three bodies. It is "the only problem that ever made my head hurt."

18th and 19th century: Celestial mechanics, the motion of the solar system, is a central question of mathematical physics. The question of the stability of the solar system receives much attention. Perturbation series are formulated by Lagrange, Laplace, and many others – the issue of stability becomes: do these series converge?

1890's: Henri Poincare understands why the problem of 3 bodies is so hard. He considers behavior of a bundle of orbits (rather than individual orbits) and discovers chaos; very complicated orbits are possible, and perturbation theory diverges.

1917: Albert Einstein understands why the old quantum theory won't work for the Helium atom (one nucleus and two electrons – a 3-body problem). It is nonintegrable so there are not invariant tori that fill phase space – can't be explained in action-angle variables. But no one pays attention.

1954: Kolmogorov (and Arnold (1962), and Moser (1962)) show that weak perturbations leave "most" invariant tori distorted but intact.

1963: Meteorologist Edward Lorenz, studying simple systems of non-linear differential equations that crudely model the atmosphere, recognizes sensitive dependence on initial conditions. He argues that long-term weather forcasting ($t \gtrsim 30\, days$) is impossible because of the "butterfly effect": whether a butterfly in Australia flaps its wings can determine whether it rains in Chicago 30 days later.

Late 1970's: Chaos finally starts to catch on, and the explosion of interest signals a paradigm shift. This is due to the convergence of a variety of factors

– Numerical experiments, made possible by increasing computational power

– Recognition that chaotic systems can have "universal" features that don't depend on the particular system that is being studied, so some things can be predicted. To some extent, chaos can be tamed.

– Recognition of relevance of chaos to a central problem: the onset of turbulent flow in (dissipative) fluids, another notoriously hard problem.

– Many laboratory experiments exhibiting chaotic behavior, and confirming some of the predicted "universal" features are performed.

1.2 Chaos and Randomness

Chaotic dynamics makes it possible for a deterministic dynamical system (like dice) to exhibit apparently "random" behavior. We would like to understand better how such "*deterministic randomness*" is possible.

To gain insight, it helps to consider the very simplest dynamical systems. We usually think of a dynamical system in terms of *continuous flow* in phase space governed by 1st-order differential equations of the form

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{F}[\mathbf{x}(t)] \tag{2}$$

Hamilton's equations are a special case.

But sometimes it is useful to consider, instead of a flow, a *map*. In effect, a map is a dynamical system with discrete time, labeled by an integer n. Stepping from time n to $n+1$, dynamical variables behave as

$$\mathbf{x}_{n+1} = \mathbf{M}(\mathbf{x}_n) \tag{3}$$

or

$$\mathbf{x}_n = \mathbf{M}^n(\mathbf{x}_o) \tag{4}$$

– the nth iteration of the map.

For a map, an initial condition determines an orbit that is a sequence of discrete points (rather than a continuous flow). The map can be continuous, though, in the sense that nearby points are mapped to nearby points

In fact, for a flow in a compact phase space, it is often possible to associate a map with the flow to capture some of its dynamical content. In an N-dimensional phase space, embed an $N-1$ dimensional surface. A one-dimensional flow generically intersects the surface at isolated points . Consider the successive points of intersection of flow and surface in which the flow pierces the surface in the same sense (with the same sign, or tangent vector to the flow). A smooth flow in N dimensions determines a *continuous, invertible* map in $N-1$ dimensions. Note that the actual time of flow elapsed between crossings of the surface will vary, so $x_n \to x_{n+1}$ does not correspond, for each n, to the same increment of time.

The map is called a *Poincare Section* of the flow. This technique is especially useful for Hamiltonian flow with 2 degrees of freedom. Then phase space is 4-dimensional but conservation of energy restricts the orbits to lie in the 3-D $H = E$ surface. The Poincare section reduces 3-D flow to 2-D map, which is easy to visualize.

The simplest maps are 1-D and nonlinear. 1-D maps already provide examples of chaotic behavior (although not when the 1-D map is the Poincare section of a Hamiltonian flow, since Hamiltonian systems with 1 degree of freedom are integrable.

Chaos means exponentially sensitive dependence on initial conditions. In Hamiltonian flow, phase space volumes are preserved, so stretching in one direction means contraction in the transverse directions.

If the $H = E$ surface is compact, the flow must fold back as it stretches, so that nearby points continue to separate while the flow remains confined.

For a 1-D map, say from unit interval to itself $I = [0, 1] \rightarrow I$ we can't have stretching for a continuous map if map is to be 1-1. (This is another reason why we can't have chaos in Hamiltonian systems with $N = 1$ – the associated Poincare map is 1-1). But we'll consider some non-invertible chaotic maps.

1.2.1 Example - The "Tent" Map

The tent map looks like this:

It stretches I by a factor of 2, and then folds it over. This is called the "tent map" because of the shape of its graph:

Suppose we iterate the tent map many times. In each iteration, sufficiently nearby points separate by a factor of two

$$(\Delta x)_n = 2\,(\Delta x)_{n-1} \qquad [(\Delta x)_{n-1} << 1] \tag{5}$$

This is an example of a *Lyapunov exponent.* The exponent h is defined by

$$(\Delta x)_n \sim e^{hn}\,(\Delta x)_o \tag{6}$$

as $n \to \infty$ (for a "typical" infinitesimal initial interval). Here we have

$$h = \ln 2 \tag{7}$$

Chaotic maps have positive Lyapunov exponents.

Iterating the tent map twice gives the map:

Iterating n times gives (see diagram):

Any initial interval eventually gets stretched over all of I, folding back and forth many times, filling I uniformly.

Closely related to the tent map is the "shift map":

We cut in half after stretching instead of folding back. This is continuous except at the point $x = \frac{1}{2}$ (where we place the cut).

Alternately, we can identify the endpoints of the interval and think of the shift map as a map from the circle to the circle:

We stretch the circle, twist it, and fold it over (a 2 to 1 map).

Iterate many times:

Takes an interval on the circle and stretches it around the circle many times. Again, the Lyapunov exponent $h = \ln 2$.

We can write the shift map as

$$\begin{aligned} x_n &= 2x_{n-1} \pmod 1 \\ &= 2x_{n-1} \quad x_{n-1} \leq \frac{1}{2} \\ &= 2x_{n-1} - 1 \quad x_{n-1} > \frac{1}{2} \end{aligned}$$

An instructive way to think about this map is to express x in base 2, as a sequence of binary digits:

$$\begin{aligned} x &= .b_1 b_2 b_3 b_4 ... \\ &= b_1 \frac{1}{2} + b_2 \frac{1}{2^2} + b_3 \frac{1}{2^3} + ... \qquad b_i = 1 \, or \, 0 \end{aligned}$$

Then under the map

$$x \to .b_2 b_3 b_4 ...... \tag{8}$$

the map "throws away" the first binary digit, and shifts all of the others one unit to the left. This is the origin of the name "shift map". (the map discards one bit of information because it is a 2 to 1 map).

This representation of the shift map makes its properties clear. We see that the map has many orbits that are periodic or "eventually periodic" – meaning that the orbit has a finite number of points. After a while the map keeps repeating itself. The initial points that lie on such periodic orbits have binary expansions that *terminate* or *repeat*. They are the rational numbers. These are a countable set, densely embedded in I.

The vast majority of real numbers in I (all but a set of measure zero) are irrational, and so do not lie on periodic orbits. Thus, the shift map provides a nice illustration of how a deterministic system can behave randomly. Suppose we can know the value of x to only some fixed accuracy (say 9 bits). Each time we iterate the map, we lose one bit of information, and we gain one new bit. But there is no way to anticipate what the new bit will be – it can be either 0 or 1, and each has probability $\frac{1}{2}$ of occuring. After 9 iterations, the initial sequence of 9 bits is completely lost. Nine new bits have replaced them, and what these nine new bits will be is *completely unpredictable*. Since we can never know the initial data precisely – eventually the result of iterating the map will be a random sequence of bits that cannot be predicted or calculated ahead of time.

"Algorithmic complexity theory" attaches a precise meaning to the notion of a random number. Imagine writing a computer program for a computer that is to generate a sequence of binary digits. For a rational number, there are of course programs that can generate the whole infinite binary expansion – e.g. if there are n digits that repeat, the program prints the same n digits over and over again. There are also irrational, even transcendental numbers $(\sqrt{2}, \pi, ...)$ for which a program of finite length generates all the digits.

But a binary sequence that cannot be the output of any program of finite length is a random sequence. New surprises keep happening as more and more bits appear.

Notice though, that the number of computer programs of finite length is *countable*. So the sequences that can be computed are a *countable set* – a subset of measure zero of I. All real numbers, except for a set of measure zero, are *random*.

The shift map, though deterministic, passes the test of randomness. If we know the first n bits of a sequence, there is now way to predict what comes next. So the orbit under the shift map is not computable. If we specify initial data to nine bits, and sample the position every nine iterations, the orbit is completely random. There is now more concise "model" of the orbit than just a table of the data, and the future cannot be computed from the past. So, the map

$$x_n = 2x_{n-1} \pmod 1 \tag{9}$$

is deterministic, and produces a random output. In a sense, the randomness doesn't come for the dynamical law; the randomness of the output comes from the randomness of the input – from the inherent randomness of the real numbers. For the real numbers are a well-defined set of mostly undefinable objects.

Does that mean that all dynamical systems produce random output – even the integrable ones? No – it is the exponential dependence on initial conditions that makes output at later times depend on more and more bits of the input. As an example of a nonchaotic map, consider

$$x_n = x_{n-1} + \beta \pmod{\mathrm{I}} \tag{10}$$

where β is a computable irrational number. Here all of the orbits are nonperiodic, but there is no chaos – nearby points stay close together. If we know x_o to m digits we can always find x_n to m digits, no matter how large m and n are. As n gets larger, we need to know β to higher accuracy, since

$$x_n = n\beta + x_o \pmod 1 \tag{11}$$

that just means we compute longer, but there is no need to specify initial data to higher accuracy – that is the typical behavior of linear dynamical systems (or of integrable ones, which become linear after a cononical transformation – e.g. to action-angle variables).

To know $x_n = n\beta + x_o$ to accuracy 2^{-m} (to m bits), we need to know

$$n\beta = 2^{\log_2 n}\beta \tag{12}$$

to this accuracy. To keep the accuracy fixed, we need to know additional digits of β at a rate growing like $\log n$ – i.e. logarithmically with the time. It is characteristic of integrable systems that computing final conditions to fixed accuracy requires computational time going like $\log t$ – the time needed to compute frequencies to higher accuracy.

For the chaotic map

$$x_n = 2^n x_o \pmod 1 \tag{13}$$

we need all of the bits of 2^n (as well as increasing no of bits of x_o) to compute to fixed accuracy – so the number of operations (powers of 2) grows linearly with the "time" – this is characteristic of chaotic systems.

The chaotic orbit is its own briefest description (since it is random) and also its own fastest computer. "Calculating" the next time step is equivalent to reading in another bit of the initial data. The dynamical system cannot really calculate the output, it can only copy the input.

The shift map is clearly ergodic. For almost any initial point, the orbit fills the interval densely and uniformly. (almost every point is random) Exceptions are the periodic points, but they are a countable set. So for almost any initial data, a time average over the orbit is equivalent to an average over I with respect to the uniform measure. For some function $A(x)$ on the interval

$$< A >_{time} = \frac{1}{n}\sum_{m=0}^{n} A\left(M^m\left(x_o\right)\right) \tag{14}$$

$$= < A >_{space} = \int_0^1 dx \; A(x) \tag{15}$$

(where $M : I \to I$ is the shift map). But the shift map actually has a stronger property than ergodicity, called *mixing*.

We can consider an arbitrary interval, and iterate the map many times – then an image of the interval fills I uniformly. Ergodic behavior means that we get a uniform distribution in phase space when we consider where the particle *has been*. Mixing behavior means that we get a uniform distribution when we consider where *it* is at a given (sufficiently late) time.

The shift map is clearly mixing since the interval keeps getting uniformly stretched around the torus over and over again.

More generally, we can consider some distribution function $\rho(x)$ on I under the map (normalize $\int_0^1 \rho(x)\, dx = 1$)

$$\rho_o \to \rho_1(x) = \int_0^1 \rho_o(y)\, \delta(x - M(y))\, dy \tag{16}$$

$$\to \rho_n = \int_0^1 \delta(x - M^n(y))\, dy \tag{17}$$

we have $\lim_{n\to\infty} \rho_n = 1$ – the uniform measure on I (which is invariant under the map). Mixing implies ergodicity, but not the other way around. Indeed

$$x_n = x_{n-1} + \beta \pmod{I} \tag{18}$$

β irrational is ergodic – every orbit fills the interval uniformly. But this map is *not* chaotic, and not mixing.

To summarize: the simple shift map has the following properties all of which are characteristic of chaotic maps

1) It is a continuous map (on the circle) with a positive Lyapunov exponent. Nearby points are stretched apart by successive iterations of the map.

2) The eventually periodic points (those lying on orbits that contain a finite number of points) are dense in the phase space, but are of measure zero.

3) Each nonperiodic orbit is ergodic; it fills the phase space uniformly – i.e. it is dense in phase space, with the probability of visiting a region tending toward an invariant measure

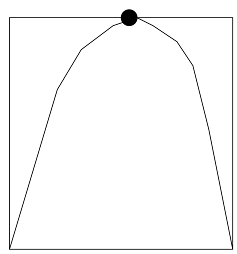
4) The map is mixing. Iterating the map causes a density on phase space to evolve toward the invariant density (at a fixed, late time).

Unlike the maps associated with Hamiltonian flows, though, the shift map is not invertible (it is 2 -to-1). By invertible we mean there is not an unique direction of flow in time that can be reversed. In fact, a 1-D map cannot be both invertible and chaotic.

Just to emphasize that the invariant measure does not have to be the uniform Lebesque measure, consider the *logistic map*

$$x_n = r x_{n-1} (1 - x_{n-1}) \tag{19}$$

(considered by Robert May as a crude model for population biology). For $r = 4$ this is a 2 -to-1 map of I to I – a kind of smoothed out version of the tent map.



In fact, we can relate it to the tent map by a change of variable

$$x = \sin^2 \frac{\pi y}{2} \tag{20}$$

$$4x (1 - x) = 4 \sin^2 \frac{\pi y}{2} \cos^2 \frac{\pi y}{2} = \sin^2 (\pi y) \tag{21}$$

so

$$\sin^2 \frac{\pi y_n}{2} = \sin^2 (\pi y_{n-1}) \tag{22}$$

or

$$\frac{\pi y_n}{2} = \pm \pi y_{n-1} + k\pi \qquad k \text{ an integer} \tag{23}$$

$$\rightarrow y_n = \pm 2 y_{n-1} + 2k \tag{24}$$

so

$$y_n = 2 y_{n-1} \qquad y_{n-1} \leq \frac{1}{2} \tag{25}$$

$$y_n = -2 y_{n-1} + 2 \qquad y_{n-1} \geq \frac{1}{2} \tag{26}$$

this is the tent map:

So $r = 4$ logistic map is "isomorphic" to the tent map – or they are "conjugate maps. We have

$$y_n = M (y_{n-1}) \tag{27}$$

where M is the tent map

$$x = g (y) \tag{28}$$

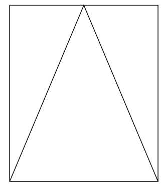
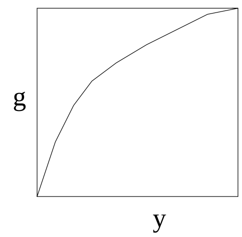


Figure 1: The tent map.



where $g(y) = \sin^2 \frac{\pi y}{2} = \frac{1}{2}(1 - \cos \pi y)$ then

$$x_{n-1} \to g\left(M\left(g^{-1}(x_{n-1})\right)\right) \tag{29}$$

thus

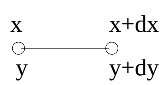
$$x_n = \tilde{M}(x_{n-1}) \tag{30}$$

where

$$\tilde{M} = g \circ M \circ g^{-1} \tag{31}$$

(with $\circ$ denoting composition of maps) corresponding to a measure on y . is a measure on x, related by

$$\tilde{\rho}(x)\,|dx| = \rho(y)\,|dy| \tag{32}$$



– probability of a point being in a interval does not depend on the coordinates that we use....

So

$$\tilde{\rho}(x) = \rho(y)\left|\frac{dy}{dx}\right| \tag{33}$$

for the map $x = \sin^2 \frac{\pi}{2} y$ we have

$$\begin{aligned} \frac{dx}{dy} &= 2\sin\left(\frac{\pi}{2}y\right)\frac{\pi}{2}\cos\left(\frac{\pi}{2}y\right) \\ &= \pi x^{1/2}(1-x)^{\frac{1}{2}} \end{aligned}$$

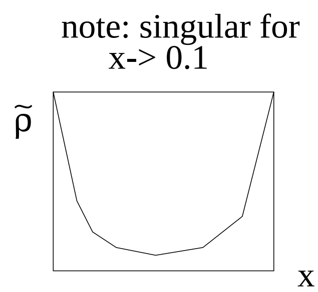
so

$$\tilde{\rho}(x) = \rho(y)\,\frac{1}{\pi\sqrt{x(1-x)}} \tag{34}$$

The invariant measure of the tent map, as for the shift map, is

$$\rho(y) = 1 \tag{35}$$

so



$$\tilde{\rho}(x) = \frac{1}{\pi\sqrt{x(1-x)}} \tag{36}$$

is the invariant measure of the $r = 4$ logistic map.

Since the tent map and $r = 4$ logistic map are conjugate ($x = g(y)$ is 1-to-1) there is a one-to-one correspondence of eventually periodic points of the two maps. So there are countably many such points for the $r = 4$ logistic map.

What happens to the Lyapunov exponent when we smooth out the map?

For

$$x_n = M(x_{n-1}) \tag{37}$$

we have

$$dx_n = M'(x_{n-1})\, dx_{n-1} \tag{38}$$

The rate at which nearby points separate (or approach) is

$$\left|\frac{dx_n}{dx_{n-1}}\right| = |M'(x_{n-1})| \tag{39}$$

For the $r = 4$ logistic map

$$M(x) = 4x(1-x) \tag{40}$$

$|M'|$ varies between 4 at endpoints and 0 at $x = \frac{1}{2}$.

At what rate do points separate on the average?

$$\frac{dx_n}{dx_o} = \frac{dx_n}{dx_{n-1}}\frac{dx_{n-1}}{dx_{n-2}} - - - \frac{dx_2}{dx_1}\frac{dx_1}{dx_o} \tag{41}$$

so

$$\left|\frac{dx_n}{dx_o}\right| = |M'(x_{n-1})|\,|M'(x_{n-2})| - - - |M'(x_o)| \tag{42}$$

The Lyapunov exponent h is defined by

$$\left|\frac{dx_n}{dx_o}\right| \sim e^{nh} \qquad \text{as } n \to \infty \tag{43}$$

so

$$h = \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n-1} \ln |M'(x_m)| \tag{44}$$

Now suppose the map is ergodic, so this time average can be expressed as an integral weighted by the invariant density. We know this is so for the $r = 4$ logistic map, since it is true for the tent map to which it is conjugate. Then the Lyapunov exponent is

$$h = \int_0^1 dx\ \tilde{\rho}(x) |\ln M'(x)| \tag{45}$$

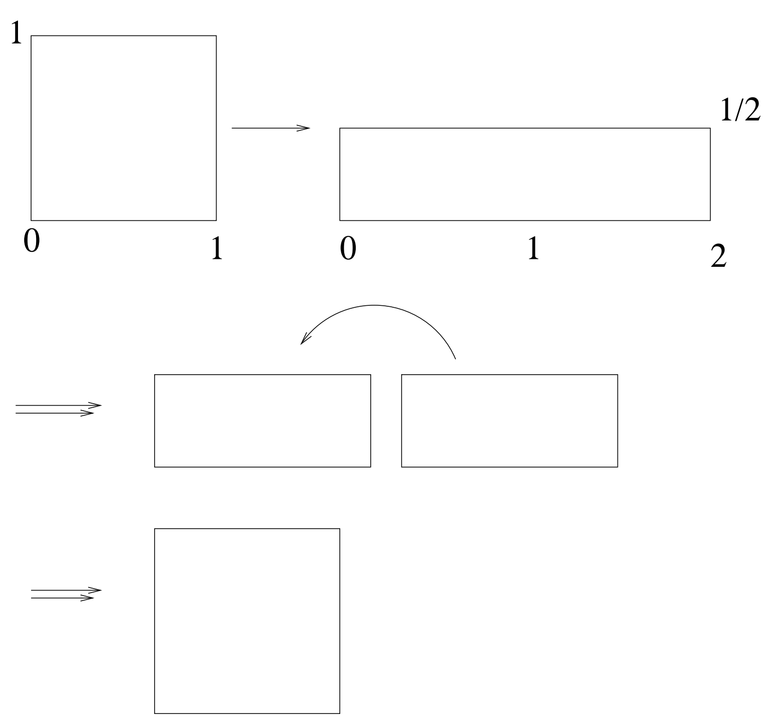
From this formula, we can show that two conjugate maps have the same h (if the isomorphosm g is differentiable). So we know

$$h = \ln 2 \quad \text{for } r = 4 \text{ logistic map} \tag{46}$$

There is a lot more to say about one-dimensional maps, and we'll return to the subject later. First, though, it will be enlightening to consider some examplesof 2-D maps. By going to 2 dimensions, we are able to construct chaotic maps that are 1-1 – in fact, they can also be *area preserving.*

The simplest example of an area preserving chaotic map is the "Baker's map" – similar to the shift map (and shares with it the property of being discontinuous on a set of measure zero). The Baker's map takes a unit square to itself: $M : I \times I \to I \times I$

First, stretch horizontally and contract vertically then cut in half and stack the right half over the left half

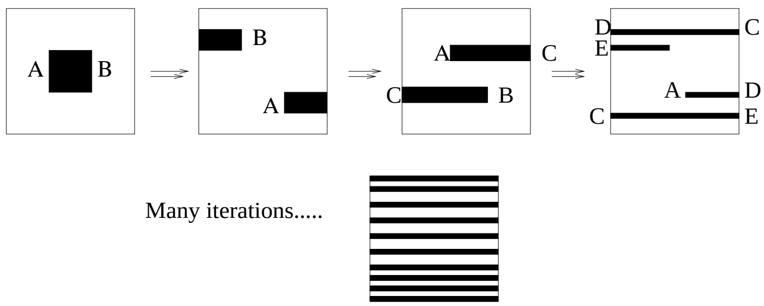


This is clearly an invertible map (unlike the shift or tent maps). Mathematically we can express this as

$$\begin{aligned} x &< \frac{1}{2}: \quad x_n = 2x_{n-1} \qquad\quad y_n = \frac{1}{2}y_{n-1} \\ x &> \frac{1}{2}: \quad x_n = 2x_{n-1} - 1 \quad y_n = \frac{1}{2}y_{n-1} + \frac{1}{2} \end{aligned}$$

This map obviously preserves the area element. Because of the cutting, it is discontinuous at $x = \frac{1}{2}$.

Suppose we iterate the map many times – stretch-cut-stack, stretch-cut-stack, over and over



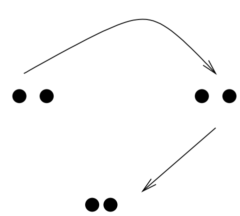
After many iterations a long, thin filament crosses the square horizontally many times.

For this map there are two distinct Lyapunov exponents – a positive one corresponding to the horizontal stretching, and a negative one corresponding to vertical contraction. It is obvious that they are

$$\begin{aligned} h_1 &= \ln 2 \\ h_2 &= -\ln 2 \end{aligned}$$

Because of the one positive Lyapunov exponent the map is chaotic.

In general, how are Lyapuov exponents in higher dimension maps to be defined? We can linearize the map near a particular orbit to find how nearby points deviate from that orbit



$$x_o \to x_1 \to x_2 \tag{47}$$

$$\Longrightarrow x_o + \varepsilon_o \to x_1 + \varepsilon_1 \to x_2 + \varepsilon_2 \tag{48}$$

$$x_1 + \varepsilon_1 = M\,(x_o + \varepsilon_o) = x_1 + \mathbf{DM}\,(x_o) \cdot \varepsilon_o \tag{49}$$

where $\mathbf{DM}\left(x_o\right)$ is the linearized map

$$\mathbf{DM}\left(x_o\right) = \begin{bmatrix} \frac{\partial M^{(1)}}{\partial x^{(1)}} & \cdots & \frac{\partial M^{(1)}}{\partial x^{(N)}} \\ .. & .. & .. \\ \frac{\partial M^{(N)}}{\partial x^{(1)}} & .. & \frac{\partial M^{(N)}}{\partial x^{(N)}} \end{bmatrix} |_{x_o} \tag{50}$$

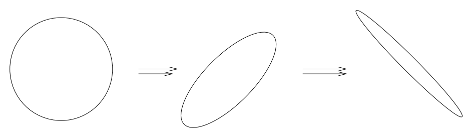
$$\mathbf{DM}^n\left(x_o\right) = \mathbf{DM}\left(x_{n-1}\right) \cdot \mathbf{DM}\left(x_{n-2}\right) ... \mathbf{DM}\left(x_0\right) \tag{51}$$

(matrix product).

The idea now is to diagonalize the matrix $DM^n\left(x_o\right)$ as $n \to \infty$. The eigenvalues behave like $\lambda_j^n \quad j = 1, ...N$. The λ_j's are the "Lyapunov numbers" and

$$h_j = \ln \lambda_j \tag{52}$$

are the Lyapunov exponents.



The sphere near x_o gets distorted into an ellipsoid after many iterations, the largest eigenvalue dominates – if it is positive there will be exponential stretching in the direction associated with the largest eigenvalue. In general, exponents can be different for different trajectories.

Like the shift map, the Baker's map has a nice symbolic representation. This time write x, y as binary expansions

$$\begin{aligned} x &= .b_1 b_2 b_3 b_4 ... \\ y &= .c_1 c_2 c_3 c_4 .... \end{aligned}$$

Now put the two sequences "back to back" by writing y backwards to the left of the decimal point:

$$...c_4 c_3 c_2 c_1 . b_1 b_2 b_3 b_4 ... \tag{53}$$

Now

$$\begin{aligned} x &< \frac{1}{2}, (b_1 = 0) \Longrightarrow x \to 2x = .b_2 b_3 b_3 b_4 b_5 ..... \\ &\Longrightarrow y \to \frac{1}{2} y = .0 c_1 c_2 c_3 ..... \end{aligned}$$

and

$$\begin{aligned} x &> \frac{1}{2}, (b_1 = 1) \Longrightarrow x \to 2x - 1 = .b_2 b_3 b_3 b_4 b_5 ..... \\ &\Longrightarrow y \to \frac{1}{2} y + \frac{1}{2} = .1 c_1 c_2 c_3 ..... \end{aligned}$$

In both cases the map just moves the decimal of the double sequence one unit to the right:

$$...c_4c_3c_2c_1.b_1b_2b_3b_4... \rightarrow ...c_3c_2c_1b_1.b_2b_3b_4b_5... \quad (54)$$

We see that the baker's map is quite similar to the shift map, and it has similar properties. But note that *we never throw away a bit* (unlike in the shift map), so this is an *invertible* map.

1) the periodic points of the map are dense in the unit square (you can demonstrate this to yourself)

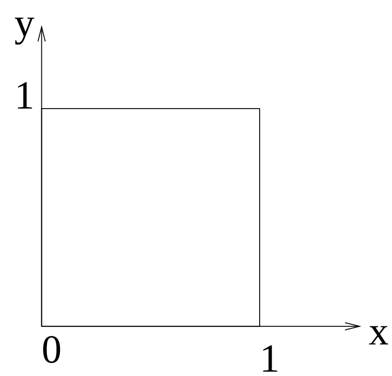
2) but for a typical point, iterating the map forward or backward eventually generates random output that is completely unpredictable – the orbits fill the square

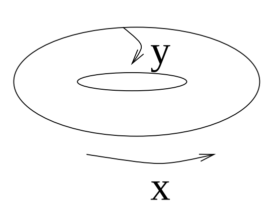
3) the map is mixing, where the invariant density is the uniform density on the unit square (as well as ergodic with all but a measure zero of points lying on orbits that uniformly fill the square).

The baker's map improves on the shift map in that it demonstrates that the above properties can hold for a map that is one to one. Next we want to see that the discontinuity in the Baker's map is not necessary.

We recall that the shift map could be construed as continuous if we regarded it as a map from the circle to the circle, i.e. periodically identified x with $x + 1$. Similarly, we can construct a continuous 2-D chaotic map on the torus – i.e. by periodically identifying x with $x + 1$ and y with $y + 1$.

The periodically identified square is topologically a product of circles $S^1 \times S^1$ or a *torus*.





The linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (\text{mod } 1) \quad (55)$$

is a continuous transformation from torus to torus if the matrix elements are integers. Then

$$\begin{pmatrix} x+1 \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (\text{mod } 1) \tag{56}$$

$$\begin{pmatrix} x \\ y+1 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (\text{mod } 1) \tag{57}$$

which ensures that, e.g. $(\varepsilon_x, \varepsilon_y)$ and $\left(1 - \varepsilon'_x, \varepsilon'_y\right)$ – nearby points on the torus – are mapped to nearby points.

This map is also area preserving if it has determinant one. A simple example is

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (\text{mod } 1) \tag{58}$$

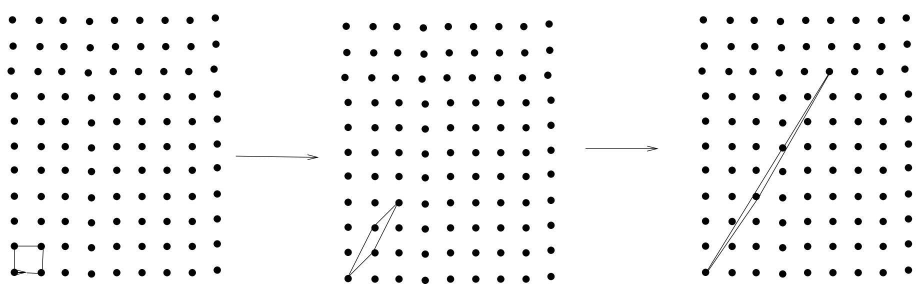
called the "cat map", inverted by

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (\text{mod } 1) \tag{59}$$

This matix has eigenvalues λ and λ^{-1}

$$\lambda = \frac{1}{2}\left(3 + \sqrt{5}\right) > 1 \tag{60}$$

–so it stretches the torus along one axis and contracts it along the orthogonal axis



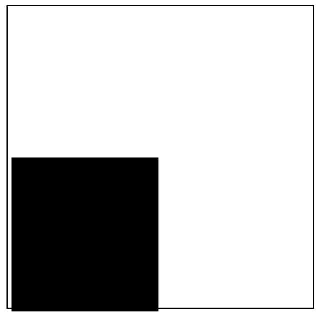
$$T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

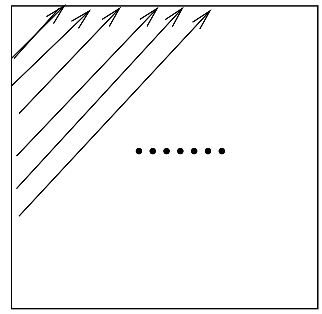
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

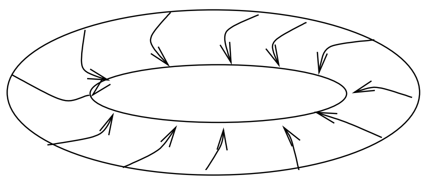
apply it twice

$$T^2 = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$







After the map acts many times the square in the lower left wraps around the square many times

–or around the torus many times

This map has the mixing property, where the invariant density is the uniform measure on the torus. Arnold called it the "cat map" because it destroys a cat who lives on the torus.

The cat map has periodic orbits; these are the fixed points of the n^{th} iteration for some n:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1 \tag{61}$$

Since the matrix has integer matrix elements, this can have solutions only when both x, y are rational. Also true, though less obvious, is that there is a solution for some n *whenever* x and y are rational. Thus, the periodic points of the cat map are (x, y) where x and y are both rational – periodic points are *dense* on the torus.

Thus the cat map provides an example of a continuous, invertible, area-preserving chaotic map on the torus. It is mixing, and its periodic points are dense.

There are also "real" dynamical systems that exhibit these properties, though this is much harder to prove. The simplest example – a torus (periodically identified square) with a circular hole removed, and a circular billiard ball that bounces elastically off of the hole. For all but a set of measure zero of initial conditions, the orbit of the center of the ball fills the regions uniformly – further, neighboring orbits diverge – the mapping is mixing as well as ergodic. The ball "forgets" the initial conditions, and *the late time behavior is random.*

1.3 Invariant Tori

We return now to the theory of integrable dynamical systems, and tie up some loose ends in our earlier discussion of action-angle variables. We will need the following theorem in the future development of chaos theory, which we state without proof (the proof can be easily sketched, however given the time I choose to skip it)

Theorem: In 2N-dimensional phase space, consider N functions of the canonical variables

$$F_1(q,p), F_2(q,p), ..., F_N(q,p) \tag{62}$$

(i) Suppose that these functions are independent (meaning that at each point the N Hamiltonian vector fields that they generate are linearly independent)

(ii) Suppose that these functions are "in involution" – the Poisson bracket of any pair vanishes

$$[F_i, F_j] = 0 \tag{63}$$

(iii) Consider the manifold $M(c_1, c_2, ...c_N)$ defined by specifying

$$\begin{aligned} F_1(q,p) &= c_1 \\ F_2(q,p) &= c_2 \\ ... &= ... \\ F_N(q,p) &= c_N \end{aligned}$$

Suppose that this manifold is *compact* (has finite volume) and is *connected* (there is a continuous curve connecting any two points), then

1. $M(c_1, ...c_N)$ is a torus. That is, there is a differentiable 1-1 mapping from a product $S^1 \times ... \times S^1$ of N circles to $M(c_1, ...c_N)$

2. Thus, if a Hamiltonian system has in involution N conserved quantities (one of which is H) fixing the values of these N quantities determines an N-torus in phase space. Any initial point on the torus remains on the torus under the Hamiltoninan flow

3. Furthermore, a cononical transformation to action-angle variables can be constructed. This transformation can be reduced to quadrature – algebraic operations and the evaluation of ordinary integrals. Thus, a dynamical system with N conserved quantities is *completely integrable.*

1.4 Canonical Perturbation Theory

We may come back to the proof of the above later if we have time. But, given the fact that dynamical systems with N conserved quantities are integrable, we whant to ask – is integrability generic? Suppose H_o is an integrable Hamiltonian. After a cononical transformation it is

$$H_o(J) \quad \text{- a function of } N \text{ action variables} \tag{64}$$

Now we perturb H_o by an infinitesimal amount

$$H(J,w) = H_o(J) + \varepsilon H_1(J,w) \tag{65}$$

where the perturbation is a function of the J's and the w's. The trajectories governed by H_o lie on invariant $N-$tori. For the nearly integrable motion governed by H, an orbit stays close to an invariant $N-$torus of H_o for a long time. But does it stay close for an infinitely long time? It may eventually wander away, over a time that gets longer as ε gets smaller.

Our strategy for addressing the question is one that has been used for centuries in celestial mechanics – it is called "time-independent perturbation theory" (see Goldstein, Chapter 11). We attempt to find, in a power series in ε, the distorted invariant tori of H that are close to the invariant tori of H_o. In other words, we try to construct, perturbatively in ε, the canonical transformation to action-angle variables for H.

Under this canonical transformation,

$$(w, J) \to (w', J') \tag{66}$$

the Hamiltonian becomes a function of the new action variables

$$H(J, w) = H'(J') \tag{67}$$

Let $G(w, J')$ be the generating function for this transformation

$$\begin{aligned} \frac{\partial G}{\partial w_i} &= J_i \\ \frac{\partial G}{\partial J_i'} &= w_i' \end{aligned}$$

To zeroth order in ε, the transformation is the identity transformation. Expanding to linear order

$$G = w_i J_i' + \varepsilon G_1(w, J') \tag{68}$$

so that

$$J_i = J_i' + \varepsilon \frac{\partial}{\partial w_i} G_1(w, J') \tag{69}$$

$$w_i = w_i + \varepsilon \frac{\partial}{\partial J_i'} G_1(w, J') \tag{70}$$

we demand that

$$\begin{aligned} H(J, w) &= H_o\left(J' + \varepsilon \frac{\partial}{\partial w} G_1\right) + \varepsilon H_1(J', w) \\ &= H'(J') \end{aligned}$$

This becomes, to linear order in ε

$$H_o(J') + \varepsilon\left(\frac{\partial}{\partial w_i} G_1\right) \frac{\partial H_o}{\partial J_i} + \varepsilon H_1 = H'(J') \tag{71}$$

Recall that

$$\frac{\partial H_o}{\partial J_i} = \dot{w}_i = \nu_i(J) \tag{72}$$

– the (unperturbed) frequency of the angle variable, so we have

$$H_o\left(J'\right) + \varepsilon \nu_i \frac{\partial}{\partial w_i} G_1\left(w, J'\right) + \varepsilon H_1\left(J', w\right) = H'\left(J'\right) \tag{73}$$

Now recall that each w_i is a periodic variable with period 1, and expand in a Fourier series

$$H_1\left(J', w\right) = \sum_{\mathbf{m}} H_{1\mathbf{m}}\left(J'\right) e^{2\pi i \mathbf{m}\cdot\mathbf{w}} \tag{74}$$

$$G_1\left(w, J'\right) = \sum_{\mathbf{m}\neq 0} G_{1\mathbf{m}}\left(J'\right) e^{2\pi i \mathbf{m}\cdot\mathbf{w}} \tag{75}$$

Here $\mathbf{m} = (m_1, m_2, ..., m_N)$ represents N *integers*. (Note that the constant $(\mathbf{m} = 0)$ term in G_1 does not contribute to $\frac{\partial}{\partial w_i} G_1$, so we are free to set it to zero).

The condition satisfied by G_1 becomes

$$H_o + \varepsilon \sum_{m} \left(2\pi i \boldsymbol{\nu} \cdot \mathbf{m} G_{1\mathbf{m}} + H_{1\mathbf{m}}\right) e^{2\pi i \mathbf{m}\cdot\mathbf{w}} = H' \tag{76}$$

We equate Fourier coefficients on both sides:

$$\mathbf{m} = 0 \quad – \qquad H' = H_o + \varepsilon H_{1\mathbf{0}} \tag{77}$$

(this determines the new Hamiltonian)

$$\mathbf{m} \neq 0 \quad – \quad G_{1\mathbf{m}} = \frac{-1}{2\pi i} \left(\frac{H_{1\mathbf{m}}}{\boldsymbol{\nu} \cdot \mathbf{m}}\right) \tag{78}$$

so we have constructed, to order ε, the canonical transformation

$$G = \mathbf{w} \cdot \mathbf{J}' - \frac{e}{2\pi i} \sum_{\mathbf{m}\neq 0} \frac{H_{1\mathbf{m}}}{\boldsymbol{\nu} \cdot \mathbf{m}} e^{2\pi i \mathbf{m}\cdot\mathbf{w}} + ... \tag{79}$$

But there is a problem. Suppose that the N frequencies are commensurate – there is a linear combination of the ν_i's with integer coefficients that vanishes. Then we have

$$\boldsymbol{\nu} \cdot \mathbf{m} = 0 \tag{80}$$

for some $\mathbf{m} \neq 0$ and one of the Fourier coefficients in the expansion of the G_1 blows up. Perturbation expansion does not make sense....

Even if the frequencies are not commensurate, there will be choices of m for which $\nu \cdot m$ is arbitrarily small. Therefore, no matter how small ε is, there will be terms in the first order correction to G that are very large. This suggests that higher order terms in ε can also be large – serious doubts are raised about the convergence of the expansion in ε of each Fourier mode of G, and also about the convergence of the Fourier expansion itself.

This is the "problem of small divisors" that plagues perturbation theory. Note that it does not arise for systems with one degree of freedom – which are know to always be integrable.

What is going on is that systems with more than one degree of freedom have resonances, and *resonance is the enemy of stability.* For example, an integrable system with $N = 2$ is like 2 uncoupled oscillators. But a generic perturbation couples these two oscillators together, however weakly. If the two oscillators have commensurate frequencies,

$$p\nu_1 - q\nu_2 = 0 \Longrightarrow q\nu_1^{-1} = p\nu_2^{-1} \tag{81}$$

then oscillator 1 completes q cycles in the time that oscillator 2 completes p cycles. When the perturbation is turned on, this means that oscillator 1 kicks oscillator 2 at regular intervals (and vice versa). Kicks add together *in phase*, and eventually accumulate to drive both 1 and 2 from their original orbits.

In spite of the problem of small divisors, perturbation theory is sometimes useful. For a typical perturbation H_1, the $H_{1\mathbf{m}}$'s are small when the $\mathbf{m}$'s are large, so we can truncate the Fourier expansion. It may be that no small divisors occur for the values of $\mathbf{m}$ that are included. In practice then, the canonical transformation can be used to accurately predict orbits for a long time.

Example: For the Sun-Jupiter-Earth system, the unperturbed frequencies are in the ration 11.8 to one. Many Fourier modes can be included before the large terms (small divisors) are encountered. Perturbation theory gives a good description of how Jupiter perturbs the motion of the Earth around the Sun, over many Earth orbits. But it doesn't tell us how the perturbation influences the Earth over an *arbitrarily large* number of orbits.

What Kolmogorov Arnold and Moser (KAM) managed to do was to find a much more powerful way of doing perturbation theory. Crudely speaking, their idea is to use an iterative procedure where the torus generated in step $n-1$ of the procedure is used as the starting point for the *nth* approximation (instead of finding the order ε^n term in the expansion about the unperturbed tori). This is roughly analogous to Newton's method for finding the zero of a function, and has much better convergence properties than an expansion in power series about the unperturbed solution.

The main results are

1) For ε sufficiently small, the improved perturbation theory really does converge, "almost always" – so most trajectories remain confined to invariant N-tori for all time

2) But all of the resonant tori with $\mathbf{m} \cdot \boldsymbol{\nu} = 0$ are destroyed! Trajectories that start out very near the resonant tori of the unperturbed system *do not* remain confined to N-tori when the perturbation is turned on.

Together these two results seem paradoxical. The resonant tori are *dense* in the phase space. If they are all destroyed, how can there be any invariant tori left? For, e.g. $N = 2$, we have the ratio

$$\frac{\nu_1(J)}{\nu_2(J)} \tag{82}$$

which varies smoothly as a function of the J_1, J_2 that label the tori. Resonant tori ($\frac{\nu_1}{\nu_2} =$ rational) are destroyed, yet "most" tori are not!

The resolution of the paradox is that the rational numbers, although dense, are countable. Hence there are sets that contain a neighborhood of each rational number, yet have arbitrarily small measure.

What KAM really showed, for $N = 2$, is that the tori survive if

$$\left| \frac{\nu_1}{\nu_2} - \frac{m}{n} \right| > \frac{K(\varepsilon)}{n^{2.5}} \tag{83}$$

for all integers m, and n, where $K(\varepsilon)$ is a function of ε (independent of m and n) that tends to 0 as $\varepsilon \to 0$ (this function was not explicitly computed by KAM). Analogous results were derived for all values of N.

The tori that are not know to survive are those with

$$\left| \frac{\nu_1}{\nu_2} - \frac{m}{n} \right| < \frac{K(\varepsilon)}{n^{2.5}} \tag{84}$$

for *some* m and n. Suppose $0 \leq \nu_1 \leq \nu_2$, so that $0 \leq \frac{\nu_1}{\nu_2} \leq 1$. How much of the total length of the unit interval is excluded by this condition? We have length

$$L \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{K}{n^{2.5}} = K \left(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \right) \sim K \tag{85}$$

(where $\left(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\right)$ is a number of order 1). (the sum estimates L since we haven't corrected for overlap of the excluded parts).

So for $K << 1$, only a small "fraction" of order K of all tori are destroyed by the perturbation.

The KAM theorem does not explicitly say what happens to the orbits in the "gaps" between the surviving invariant tori. It does indicate that for ε fixed and small, these gaps are found near th resonant tori of the unperturbed problem with

$$\frac{\nu_1}{\nu_2} = \frac{m}{n} \tag{86}$$

and are most prominent for small values of n.

Thus, the structure of the orbits for a slightly perturbed integrable system is remarkably intricate. The region of phase space in which tori are destroyed has a small volume, but is densely distributed in phase space! No wonder perturbation theory is hard!

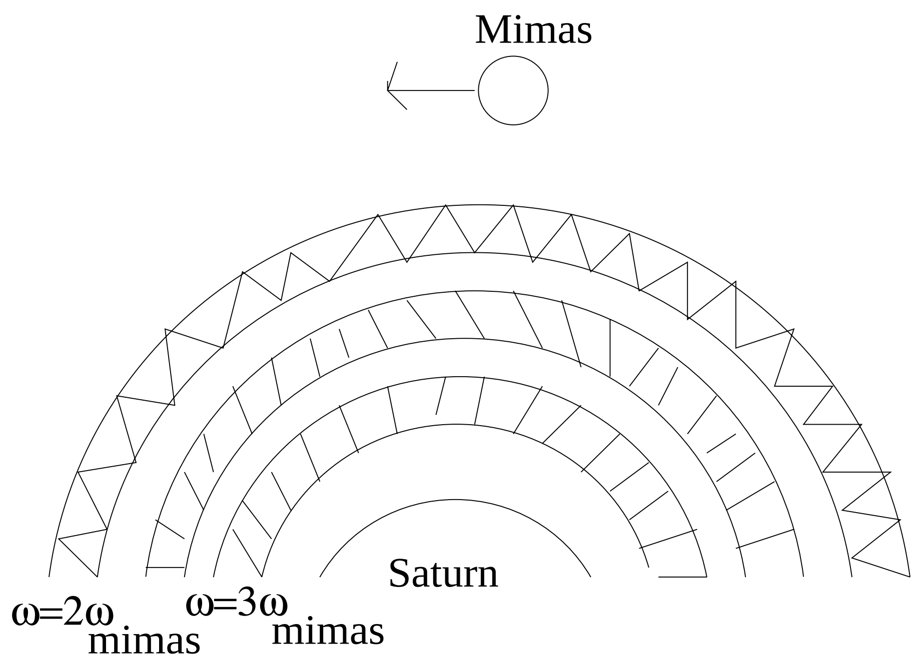
In practice, the high order gaps ($n >> 1$) may be very hard to observe, since they are very narrow. They are likely to be hidden by noise (random uncontrolled perturbations) or destroyed by dissipative effects.

A nice example of KAM gaps is provided by the ring system of Saturn.

The rings are composed of rocks moving in circular orbits around Saturn. There are conspicuous gaps in the rings that are clear of particles.

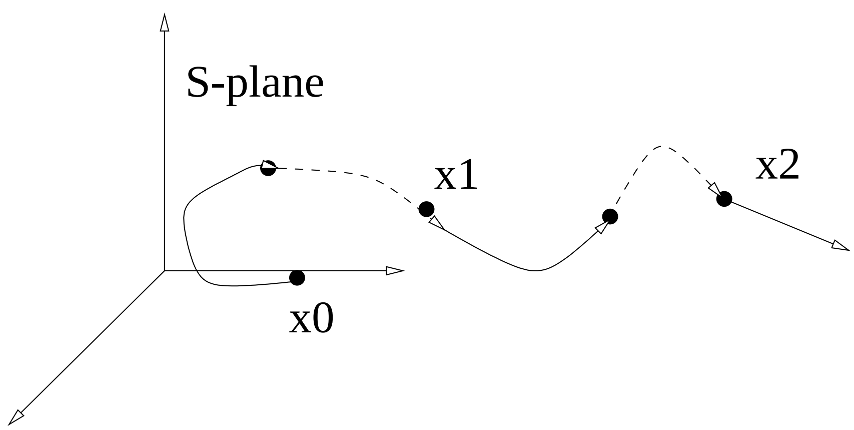
The gaps occur where the orbits are in resonance with the orbits of various satellites of Saturn. For example, two of the most conspicuous gaps occur where the orbits have frequency twice or three times the frequency of the moon Mimas.

You may be aware that there are many examples in the solar system of frequencies that are locked *into* resonance, contrary to KAM expectations. For example, the period of Mimas is $\frac{1}{2}$ the period of the satellite Tethys. The most famous example is the coincidence of the revolution and rotation periods of Earth's moon These coincidences are consequences of dissipative effects ("tidal friction") and so cannot be explained by Hamiltonian evolution.



1.4.1 The Fate of Resonant Tori

Now we want to try to understand what happens, for a slightly perturbed integrable system, in the gaps between the invariant tori (that exist according to the KAM theorem). We study this using the Poincare section technique. For ease of visualization, we specialize to the case of $N = 2$ (two degrees of freedom). The condition $H = E = const$ determines a $3 - D$ submanifold of the $4 - D$ phase space.

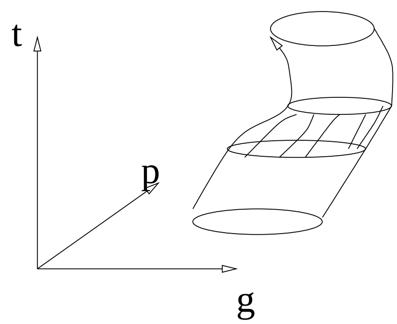


Now choose a $2 - D$ surface S embedded in the $H = E$ manifold. The orbits of the Hamiltonian system intersect S at isolated points. Successive points where the orbit pierces through S in the same sense are successive iterates of the map. (Remember the time interval associated with map points varies with the point X in S). This map

$$M : S \to S \tag{87}$$

is the *Poincare map.*

The Poincare map has a very important property – it is an area-preserving $2-D$ map. This follows from the Poincare-Cartan Theorem. Recall that for a "tube of trajectories in extended phase space, we have that



$$\oint_{\gamma_1} (p_i dq_i - H dt) = \oint_{\gamma_2} p_i dq_i - H dt \tag{88}$$

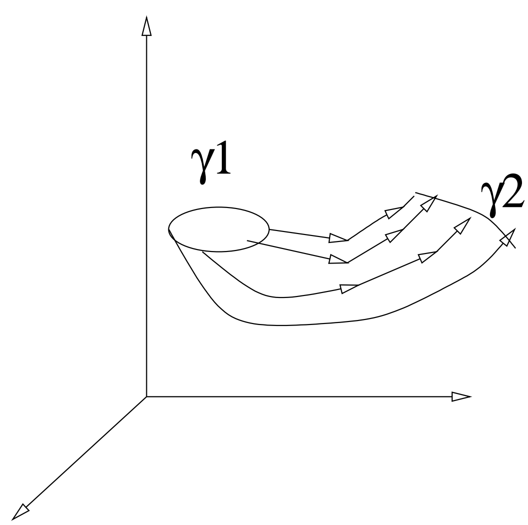
for any two closed paths γ_1 and γ_2 in extended phase space that enclose the tube. If this tube is contained in the surface with $H = E = const$, we have

$$\oint_{\gamma} H dt = E \oint_{\gamma} dt = 0 \tag{89}$$

so Poincare-Cartan becomes

$$\oint_{\gamma_1} p_i dq_i = \oint_{\gamma_2} p_i dq_i \tag{90}$$

Now choose γ_1 and γ_2 to be intersections of the tube of trajectories with the surface S.

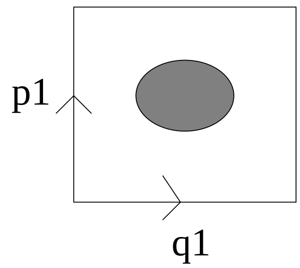


The the above says that phase space area is preserved by the map.

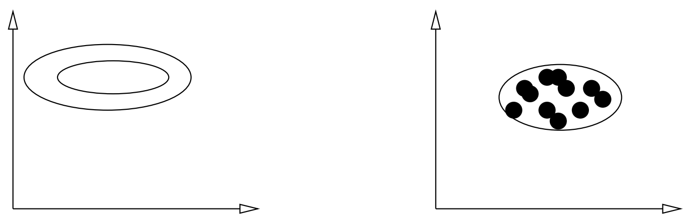
$$\oint_{\gamma} p_i dq_i = \int_{\sigma} dp_i dq_i \tag{91}$$

e.g. if we choose S to be the surface $q_2 = 0$, and parameterize S by q_1,p_1 this area element is

$$\oint p_1 dq_1 \tag{92}$$



By studying the Poincare map, we can tell whether the system is integrable. Invariant tori will generically intersect the surface S (with a particular sense) on a closed curve. So orbits of the Poincare map are typically dense on a closed curve, for integrable systems. If the system is ergodic, the orbit will tend to fill S. More generally, if the orbit of the Hamiltonian system is not confined to an invariant torus, the orbit of the Poincare map will tend to fill a two-dimensional region.



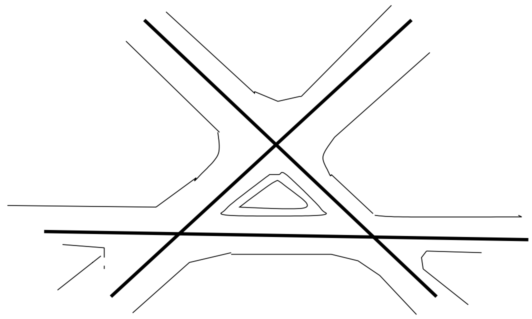
1.4.2 Example: Henon-Heiles Potential

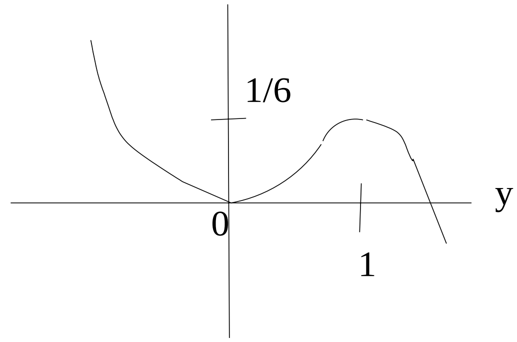
As an example of a nonintegrable system, consider a particle in two-dimensions moving in a potential

$$\begin{aligned} H &= \frac{1}{2}\left(p_x^2 + p_y^2\right) + V \\ V &= \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3 \end{aligned}$$

In polar coordinates:

$$\begin{aligned} x^2 y - \frac{1}{3}y^3 &= r^3\left(\cos^2\theta\sin\theta - \frac{1}{3}\sin^3\theta\right) \\ &= \frac{1}{3}r^3\left(\left(\cos^2\theta - \sin^2\theta\right)\sin\theta + 2\sin\theta\cos\theta\cos\theta\right) \\ &= \frac{1}{3}r^3\sin 3\theta \end{aligned}$$





so

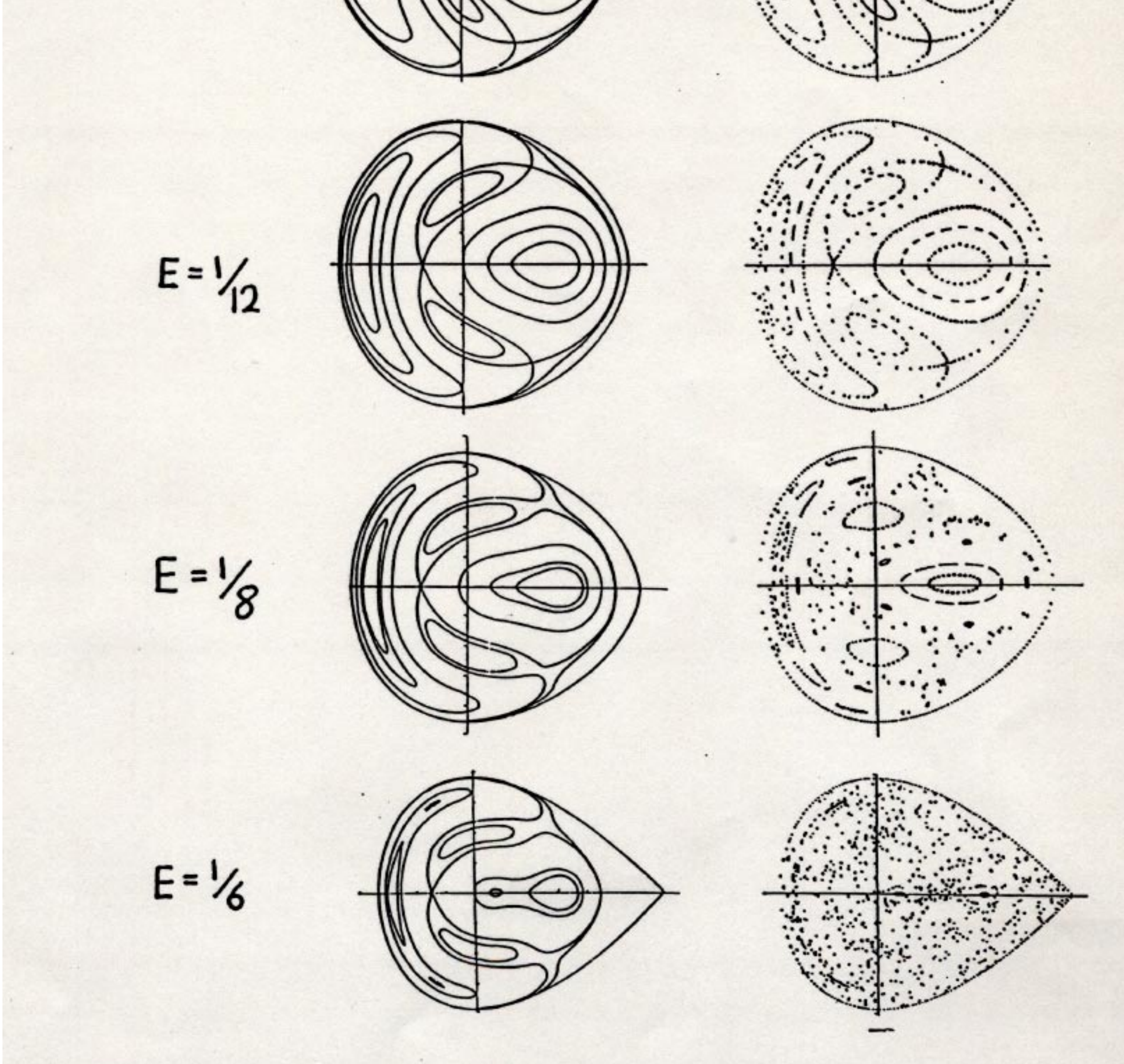
$$V = \frac{1}{2}r^2 + \frac{1}{3}r^3 \sin 3\theta \tag{93}$$

Equipotentials look as shown.

There is a local minimum at $r = 0, V = 0$. On the line $x = 0$, $V = \frac{1}{2}y^2 - \frac{1}{3}y^3$ – a local maximum occurs at $y = 1, V = \frac{1}{6}$ Also, $V = \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$ is independent of x on the line $y = -\frac{1}{2}$, $V = \frac{1}{6}$

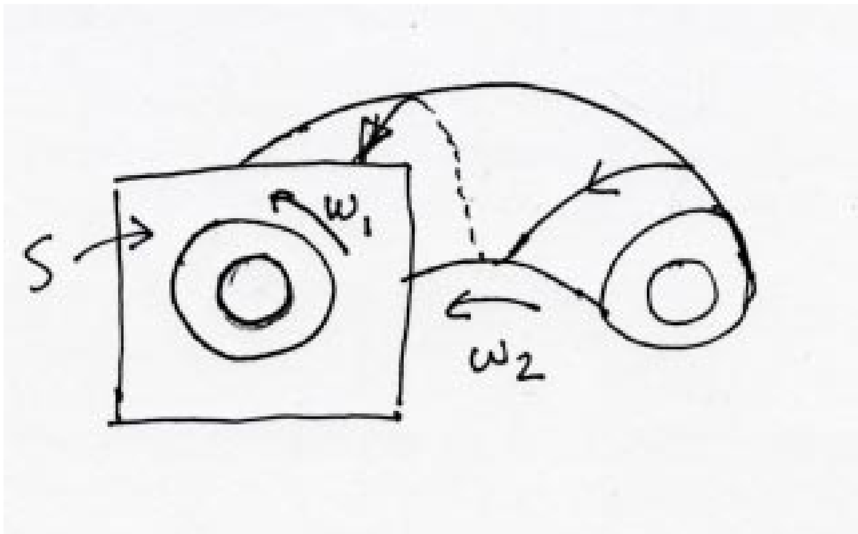
From 3-fold rotational symmetry, we can infer he shape of equipotentials. The energy surface is bounded only for $E < \frac{1}{6}$. For $0 < E << \frac{1}{6}$ the potential is very nearly harmonic, and the cubic term can be regarded as a small perturbation.

The figure shows numerical calculations of the Poincare map for this system (right panel). These are orbits of the map, shown in the x, p_x plane. On the left are the perturbed tori found in perturbation theory, carried out to eighth order. For $E \lesssim \frac{1}{9}$ the orbits seem to lie on invariant curves – those predicted by the perturbation theory. But for $E = \frac{1}{8}$ and $E = \frac{1}{6}$ the destruction of the tori is apparent. In fact, all of the points for $E = \frac{1}{8}$ and $E = \frac{1}{6}$ were generated by a single trajectory.


E = 1/24
E = 1/12
E = 1/8
E = 1/6

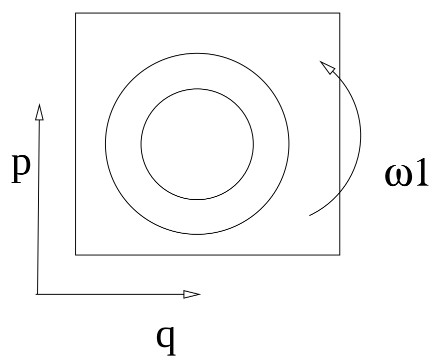
1.4.3 Perturbing the Twist Map

Continue to consider the $N = 2$ case. We'll examine in more detail what happens to the resonant tori when an integrable system is perturbed.



Suppose we have found action-angle variables for an integrable system: J_1, J_2, w_1, w_2 . We can choose the surface S that is used to construct the Poincare map to be the surface $w_2 = 0$. Then we may use as coordinates on S the variables w_1, J_1 (J_2 is fixed by the condition $H(J_1, J_2) = E$, once we specify J_1. So in the space $H = E$, there are *nested tori* labeled by J_1).

In S, the tori become concentric circles



$\theta = 2\pi w_i$ is the angular coordinate in the plane. We may choose J_1 to be the "radial coordinate"

$$J_1 = \oint pdq \to 0 \tag{94}$$

at the "origin" where the circle shrinks to zero. Or, if we want ρ to be the usual radial coordinate, with $\pi\rho^2$ being the area of a circle of radius ρ: define ρ by $J = \oint pdq = area = \pi\rho^2$.

What is the Poincare map (for the integrable system)? In angle variables, the trajectories are

$$\begin{aligned} w_1 &= \nu_1 t + const \\ w_2 &= \nu_2 t + const \end{aligned}$$

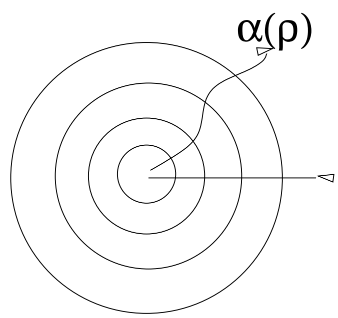
so the orbit returns to $w_2 = 0$ after $\Delta t = \nu_2^{-1}$ elapses. Thus

$$\Delta w_1 = \frac{\nu_1}{\nu_2} \equiv \alpha\left(J_1\right) = \alpha\left(\rho\right) \tag{95}$$

or

$$\Delta\theta = 2\pi\alpha\left(\rho\right) \tag{96}$$

This is a "twist map" – each circle (of constant radius ρ) is rigidly rotated by teh angle $2\pi\alpha$ – but with angle of rotation depending on the radius ρ



The KAM theorem can be expressed as a statement about perturbations of the twist map. The unperturbed twist map T is

$$\begin{aligned} T &: \quad \rho_{n+1} = \rho_n \\ \theta_{n+1} &= \theta_n + 2\pi\alpha\left(\rho_n\right) \end{aligned}$$

A generic small perturbation of the map has the form

$$\begin{aligned} T &: \quad \rho_{n+1} = \rho_n \ + \varepsilon f\left(\rho_n, \theta_n\right) \\ \theta_{n+1} &= \theta_n + 2\pi\alpha\left(\rho_n\right) + \varepsilon g\left(\rho_n, \theta_n\right) \end{aligned}$$

KAM say that, for $\varepsilon << 1$, most points in the plane lie on invariant curves of the perturbed map. Only the circles for which α is "sufficiently close to rational" are in danger of being destroyed. This special ($N = 2$) case of the KAM theorem applies to any sufficiently smooth *area preserving* perturbation.

Resonant tori of the unperturbed system are those for which $\alpha\left(\rho\right)$ is a rational number. Suppose

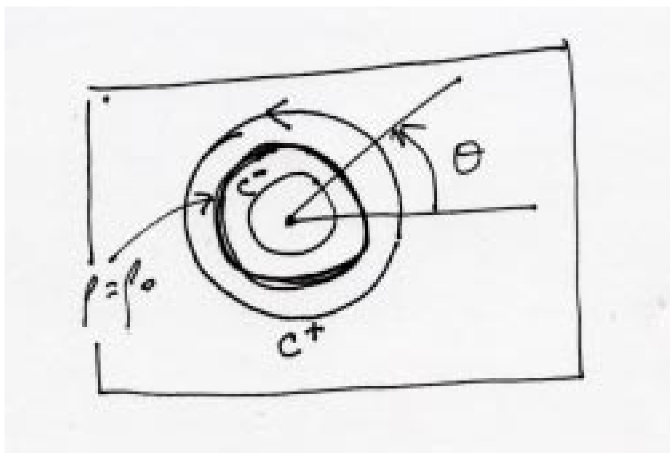
$$\alpha\left(\rho_o\right) = \frac{r}{s} \tag{97}$$

where r and s are integers that are relatively prime (no common factors). Then all points on the circle of radius ρ_o lie on finite periodic orbits of the twist map T. Points on the orbit are

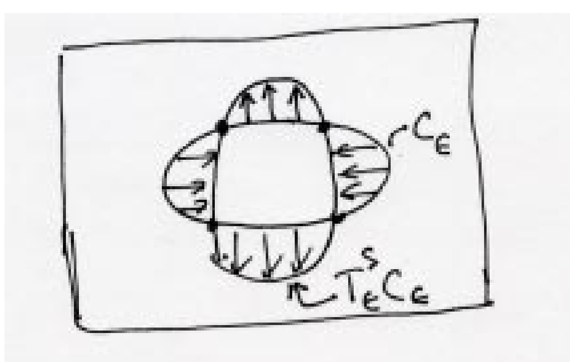
$$\theta_o, \quad \theta_o + 2\pi\frac{r}{s}, \quad \theta_o + 2\pi\frac{2r}{s}, \quad ......, \quad \theta_o + 2\pi\frac{(s-1)\,r}{s} \tag{98}$$

–(orbit of length s). Every point on this circle is a fixed point of the map T^2– which rotates the circle by $2\pi r$. Now imagine turning on the perturbation, and consider T_ε^2. What happens to the circle of fixed points of T^s when $\varepsilon > 0$?

The generic perturbation removes most of these fixed points, but not all of them. In fact, we can show that, generically, the number of fixed points that survive is *an even (integer) multiple of* s. This is called the Poincare-Birkhoff fixed point theorem.



To see this, first consider the unperturbed map again. Generically $\alpha'(\rho) \neq 0$ at $\rho = \rho_o$. This means that, if we consider circles C_+ with ρ slightly larger than ρ_o and C_- with ρ slightly smaller, the map T^s (which leaves the $\rho = \rho_o$ circle fixed) rotates C_+ slightly one way and C_- slightly the other way. If ε is very small, this means that for each value of the polar angle θ, there is a point between C_- and C_+ that is mapped radially by T_ε^s. Thus there is a closed curve C_ε that is mapped radially by T_ε^s, and approaches the $\rho = \rho_o$ circle as $\varepsilon \to 0$.



Now, we remember that the map T^s is area preserving. Therefore, the closed curves C_ε and $T_\varepsilon^s C\varepsilon$ enclose the same area. This means that C_ε and $T_\varepsilon^s C\varepsilon$ must intersect. Generically (excluding the possibility that the two closed curves are tangent somewhere, which can happen only at isolated values of ε) the two two curves must intersect in an even number of points. These points of intersection are the fixed points of T_ε^s.

In fact, the number of fixed points must be a multiple of s. Suppose x_o is a fixed point of T_ε^s. Then $x_o, T_\varepsilon x_o$, $T_\varepsilon^2 x_o, ...T_\varepsilon^{s-1} x_o$ are s distinct points – a closed orbit of T_ε that becomes the order s orbit of T' as $\varepsilon \to 0$. Each of these points is a fixed point

$$T_\varepsilon^s \left(T_\varepsilon^m x_o\right) = T_\varepsilon^m T_\varepsilon^s x_o = T_\varepsilon^m x_o \tag{99}$$

Since the number of fixed points is even and a multiple of s, it is an even multiple of s if s is odd. We will see momentarily that the number of fixed points must be an even multiple of s, even if s is even.

1.4.4 Classification of Fixed Points

Next we want to consider the issue of the stability of the fixed points. If we imagine iterating T_ε^s many times, we want to know whether points close to the fixed oint stay on orbits that remain close to the fixed point, or whether they are eventually driven away.

We study stability by *linearizing* the map near the fixed point. For a two-dimensional map

$$\begin{aligned} M &: \\ x_1 &\to M_1\left(x_1, x_2\right) \\ x_2 &\to M_2\left(x_1, x_2\right) \end{aligned}$$

the linearized map at x_o is the 2×2 matrix

$$\mathbf{DM}\left(x_o\right) = \begin{pmatrix} \frac{\partial M_1}{\partial x_1} & \frac{\partial M_1}{\partial x_2} \\ \frac{\partial M_2}{\partial x_1} & \frac{\partial M_2}{\partial x_2} \end{pmatrix} |_{x=x_o} \tag{100}$$

Then

$$\begin{aligned} M &: \quad x_o + \delta x \to M\left(x_o + \delta x\right) \\ &= \quad M\left(x_o\right) + \mathbf{DM} \cdot \delta x = x_o + \mathbf{DM} \cdot \delta x \end{aligned}$$

or

$$M : \delta x \to \mathbf{DM} \delta x \tag{101}$$

If M is an area-preserving map, then

$$\det\left(\mathbf{DM}\right) = \pm 1 \tag{102}$$

(-1 if the map is orientation-reversing). [In fact, for the Poincare map of a Hamiltonian system, the linearized map has determinant 1 – it is a sympletic matrix, as a consequence of the symplectic structure that is preserved by Hamiltonian time evolution.]

The linearized map has two eigenvalues, and the product of the eigenvalues is 1 (since $\det(\mathbf{DM}) = 1$). Now we distinguish two cases:

1) Complex eigenvalues ($\text{Im}\,\lambda \neq 0$)

The eigenvalues come in a complex conjugate pair $e^{i\beta}, e^{-i\beta}$. There is a (not necessarily orthogonal) basis in which **DM** is a *rotation* by the angle β. In this basis, it is clear that the invariant curves preserved by iterations of **DM** are circles about the fixed point. Hence the fixed point is stable.

In the original basis, we have

$$\mathbf{DM} = \mathbf{A}\ \mathbf{R}(\beta)\ \mathbf{A}^{-1} \tag{103}$$

where $\mathbf{R}(\beta)$ is the rotation by β – so **DM** differs from $\mathbf{R}(\beta)$ by a similarity transformation.

Recall that a general linear transformation takes a circle to an ellipse: if

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{104}$$

then $R^2 = x^2 + y^2$ becomes $R^2 = (ax+by)^2 + (cx+dy)^2$. We can rediagonalize this quadratic form to obtain

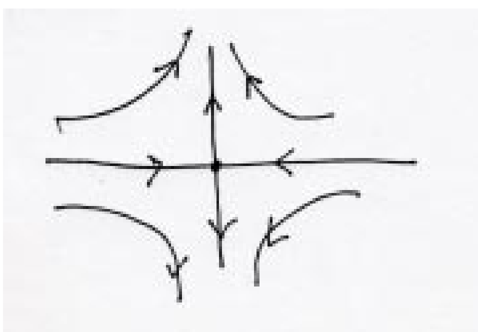
$$I = \frac{(x-c_1)^2}{a_1^2} + \frac{(y-c_2)^2}{a_2^2} \tag{105}$$

Thus, in general, the invariant curves of a linear transformation with eigenvalues $e^{\pm i\beta}$ are ellipses:

($\mathbf{A}^{-1}$ takes ellipse to a circle, which is preserved by $R(\beta)$, and then is mapped back to the ellipse by **A**.)

For this reason, stable fixed points of an area preserving map are called "*elliptic fixed points*".

2) Real Eigenvalues The two eigenvalues are λ, λ^{-1}. Suppose $|\lambda| > 1$. There is a



(not necessarily orthogonal) basis in which the transformation is

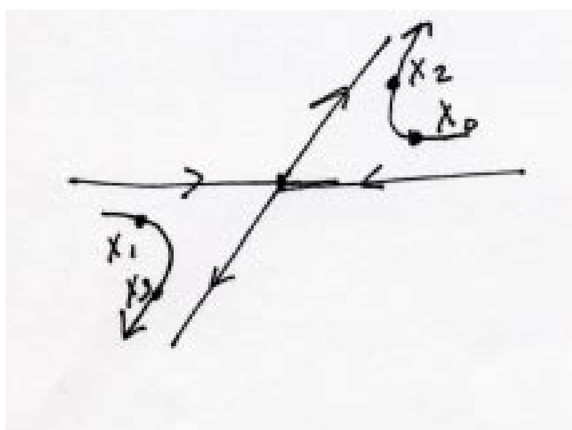
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{106}$$

The invariant curves are *hyperbolas* ($xy = \text{const}$)

For $\mathbf{DM} = \mathbf{A} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mathbf{A}^{-1}$, then the invariant curves are still hyperbolas centered at the fixed point because a linear transformation maps a hyperbola to a hyperbola – except the asymptotes need not be orthogonal:

This type of fixed point is called *hyperbolic*. It is clearly unstable – along the eigenvector with eigenvalue $|\lambda| > 1$, nearby points exponentially diverge from the fixed point, with separation increasing like $|\lambda|^m$ after m iterations.

[There are two types of hyperbolic fixed point: either $\lambda > 1$ or $\lambda < -1$. In the latter case, the orbit hops back and forth between the two branches of the hyperbola. We then speak of a "hyperbolic fixed point with reflection".]

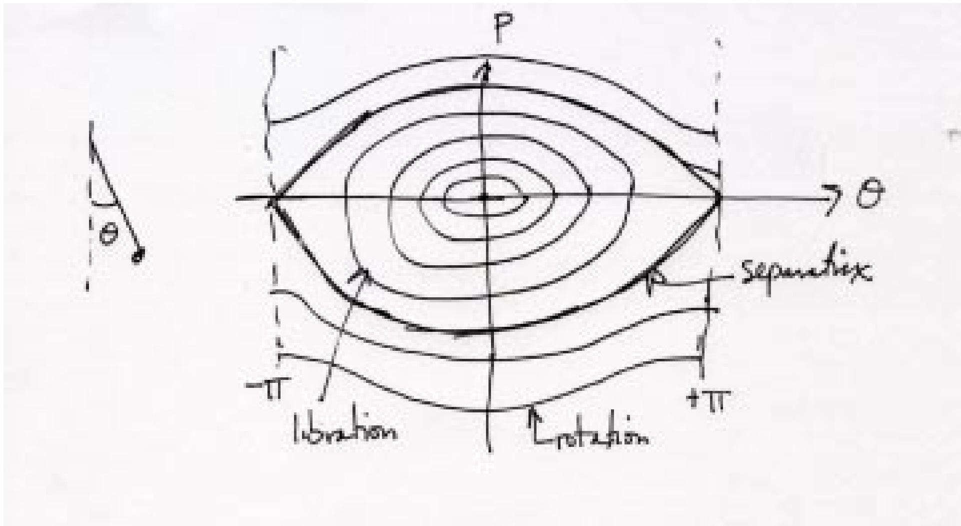


On the boundary between the elliptic and hyperbolic cases is the case $\lambda_1 = \lambda_2 = \pm 1$. Then there is a basis in which

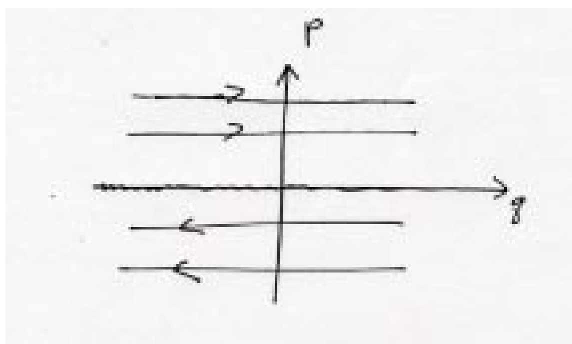
$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 & C \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{107}$$

(if $\lambda = +1$). The invariant curves are the lines y=constant (with $y = 0$ corresponding to a line of fixed points). In this case we say the fixed point is parabolic. Parabolic fixed points are non-generic and become elliptic or hyperbolic when slightly perturbed.

Examples of fixed points of all three types occur in $N = 1$ *flow* diagrams for the pendulum:



Elliptic fixed point at $p = \theta = 0$. Hyperbolic fixed point at $p = 0, \theta = \pi$
For the free particle:



Line of parabolic fixed points at $p = 0$.

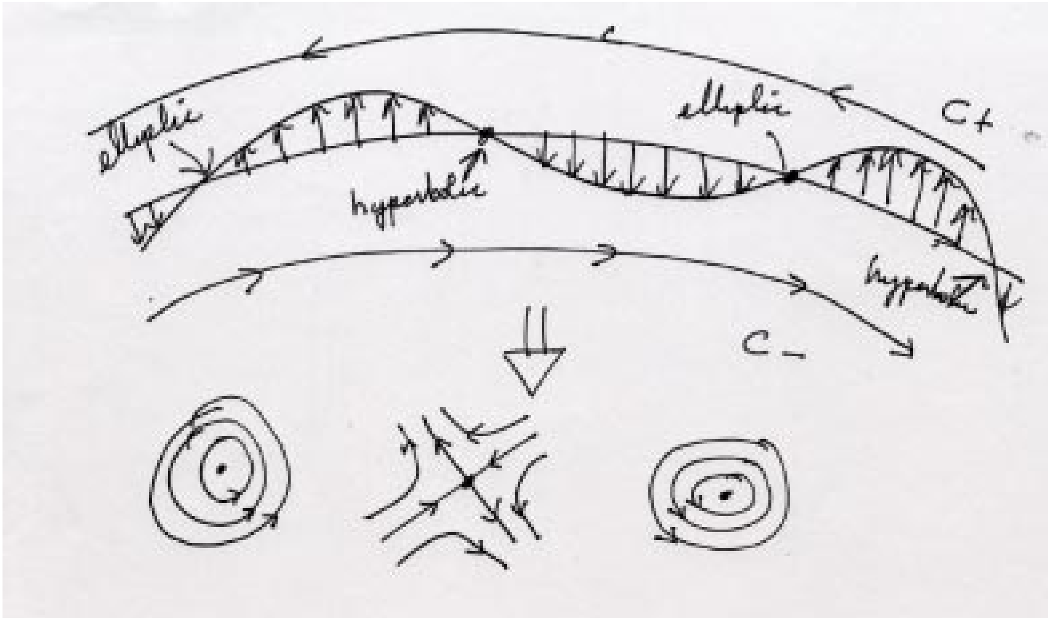
Now we return to our discussion of the Poincare-Birkhoff theorem. We saw that T_ε^s has fixed points. Are these fixed points elliptic or hyperbolic?

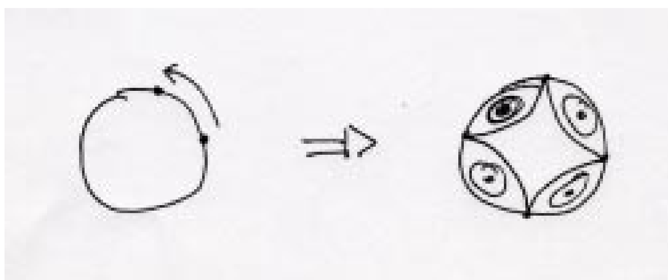
By considering the geometry of the orbits near the fixed points, we can see that the type of fixed point alternates as we travel along the radially mapped curve C_ε : elliptic-hyperbolic-elliptic-hyperbolic..... Thus, of the fixed points of T_ε^s, generically half are elliptic and half are hyperbolic. If x_o is an elliptic (or hyperbolic fixed point of T_ε^s, then so is $T_\varepsilon^m x_o$. This is why the number of fixed points is $2ks$ (k = integer), because both the number of elliptic fixed points and the number of hyperbolic fixed points is a multiple of s.

So if T rotates a circle C by the angle $2\pi\frac{r}{s}$, the orbits of the perturbed map near C break up into an "island chain" with ks islands (where typically $k = 1$).

There are s elliptic fixed points, and T_ε carries a point near one of the fixed points to a point near a fixed point r steps along in the chain. And there is a hyperbolic fixed point between each pair of elliptic fixed points.

Thus, the gap between the surviving invariant KAM tori breaks up into smaller tori. But we can now apply the KAM theorem and the Poincare-Birkoff theorem to these smaller





tori. The irrational small tori are stable, but the small perturbations from the linear approximation to the map near the elliptic fixed point cause the rational small tori to break up further into elliptic (and hyperbolic) fixed points surrounded by even smaller tori. This process continues indefinitely, down to arbitrarily small scales. The resulting structure of the orbits in phase space – with invariant tori of all sizes filling the gaps between larger invariant tori looks like this:

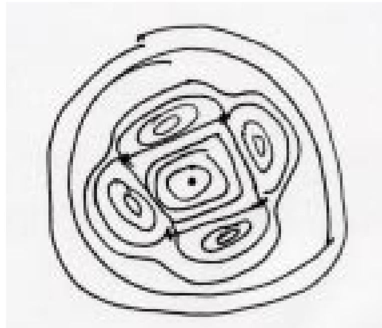
1.4.5 The Homoclinic Tangle

To complete the picture, we must understand the behavior of the orbits in the vicinity of the hyperbolic fixed points.

Associated with a hyperbolic fixed point P of a map T are the stable manifold $M_+(P)$ and the unstable manifold $M_-(P)$ of the fixed point. These are defined by

$$\begin{aligned} M_+(P) &= \{x| \lim_{n\to\infty} T^n(x)\} = P \\ M_-(P) &= \{x| \lim_{n\to-\infty} T^{-n}(\}x) = P \end{aligned}$$

Points on the stable manifold approach P as the map is iterated forward in "time". Points on the unstable manifold approach P as the map is iterated backward in "time".



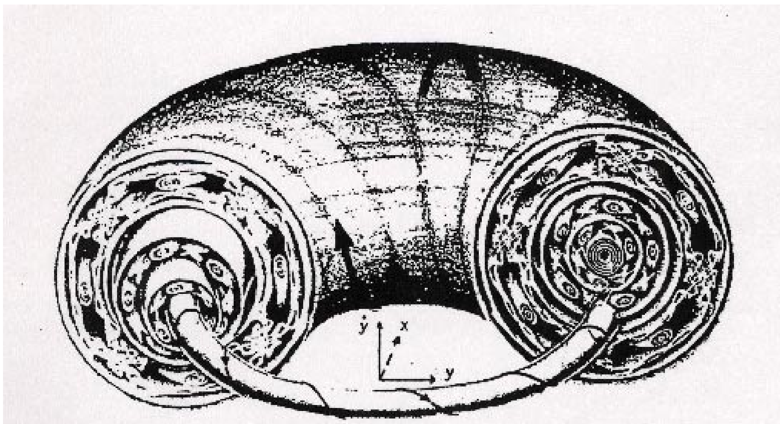
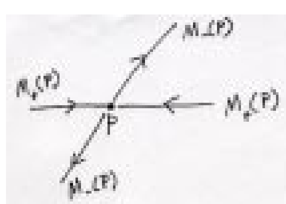


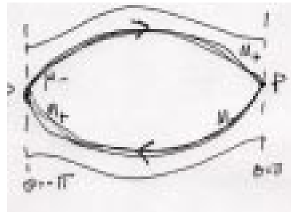
Figure 11
Nested K.A.M. tori in the 3-dimensional x,y,ẏ surface of constant energy of a Hamiltonian system like the Hénon-Heiles system (2.8-10); see figs. 10 and 4 (taken from [16]). Note the smooth regular K.A.M. tori about the center and about the elliptic orbits which, together with the chaotic regions, populate the gaps between the K.A.M. tori A magnification about such an elliptic orbit yields the same picture all over, *etc. ad infinitum*.

In the linearized approximation near the fixed point, for an area-preserving 2-D map, the stable manifold is the line determined by the eigenvector with eigenvalue $|\lambda| < 1$, and the unstable manifold is the line determined by the eigenvector with eigenvalue $|\lambda| > 1$.

Now lets try to follow $M_+(P)$ and $M_-(P)$ beyond the region where the linearized approximation applies. For an integrable system the unstable manifold of one fixed point typically matches up smoothly with the stable manifold of another fixed point (or perhaps the same fixed point).

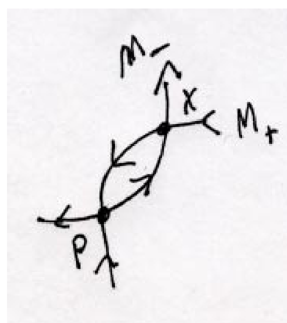
Recall, again, the pendulum. Here $M_+(P) = M_-(P)$ the orbit that leaves $P = (\theta = \pi, p = 0)$ in the unstable direction returns to P along the stable direction. More generally:





the orbit that leaves one fixed point arrives at another fixed point.

This smooth joining of stable and unstable manifolds is non-generic. It does not occur in nonintegrable systems. The generic behavior



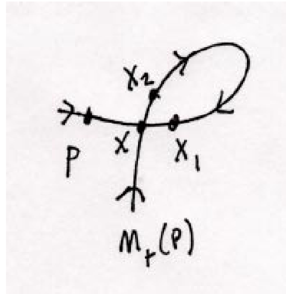
is that the stable manifold of P and its unstable manifold cross one another at a point of intersection, called a "homoclinic point", or $M_+(P)$ intersects the unstable manifold of another fixed point, in which case the point of intersection is called a "heteroclinic point".

This simple observation leads to a remarkably complex picture of the structure of the orbits. We can show that $M_+(P)$ cannot intersect itself, nor can $M_-(P)$ intersect itself. But if $M_+(P)$ intersects $M_-(P)$ in one point (as it typically does) then there must be an infinite number of points of intersection! (The existence of one homoclinic point implies an infinity of others.) This means that there must be an exceedingly complex stretching, bending and folding of the phase space as the map is iterated – in other words: the motion in the gaps between tori is chaotic.

$M_+(P)$ **cannot intersect itself:**

Suppose it does, at the point x. Then there are two points $x_{1,2}$ on $M_+(P)$ that are very close to x in phase space, but "far apart" on $M_+(P)$, as shown.

After we iterate the map a few times, the images of x_1 and x_2 are close to P, but the image of the finite arc between x_1 and x_2 lies between the images of x_1 and x_2. This means that we can let x_1 and x_2 get arbitrarily close to one another, while their images stay a finite



distance apart. This contradicts the continuity of the map, and so is not possible. Same goes for $M_-(P)$.

But $M_+(P)$ and $M_+(P)$ can intersect

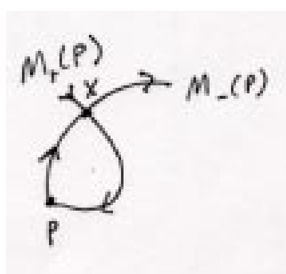
This would not be possible for a continuous flow but it is allowed for a map. If we interpret the diagram as a flow diagram, then the orbit beginning at x would not be unique. But for a map, the image of x does not have to be near x.

What we can say, though, is that if x is a homoclinic point, then so are Tx and $T^{-1}x$. Both of these points have the property that they approach P if T is iterated forward or backward, because x has that property. Furthermore, each point

$$T^n x, \quad -\infty < n < \infty \tag{108}$$

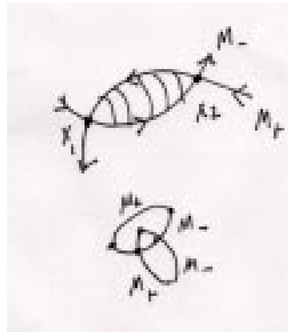
is a distinct point (otherwise x would be a periodic point of T or T^{-1}, and then $T^n x$ could not approach P as $n \to \infty$ and $n \to -\infty$). So the $T^n x$ are an infinite number of homoclinic points that accumulate at P.

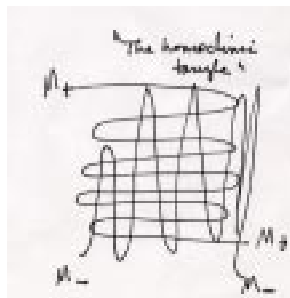
An even stronger statement: Between each pair of homoclinic points on $M_-(P)$ (or $M_+(P)$) is another homoclinic point (so homoclinic points are actually dense in M_- and M_+). To see this, suppose x_1 and x_2 are two successive homoclinic points on M_-. Consider the (shaded) are enclosed by the arcs of M_+ and M_-,



bounded by the points x_1 and x_2. Let T^n act on this region. Since T is area preserving, and only a finite area of phase space is available (intersection of $H = E$ manifold with the Poincare surface S is compact) the image under T^n and T^m must overlap for some n and m. This means there is an intersection of M_+ and M_- inbetween x_1 and x_2, contrary to assumption.

So M_+ and M_- must stretch and fold, stretch and fold,...... Orbits in the region of phase space containing the hyperbolic fixed points are *area-filling*, and do *not* lie on smooth invariant curves.





To summarize, for the generic perturbed integrable system, we have a wonderfully complex picture: each resonant torus breaks into elliptic and hyperbolic fixed points. The hyperbolic fixed points are surrounded by chaotic orbits. The elliptic fixed points are surrounded by invariant (irrational) curves that break up further, and repeat the whole structure ad infinitum. *A truly astonishing compromise between "integrable" and "ergodic" behavior!*

Note: for $N = 2$, the surviving invariant tori are impenetrable strata that confine the chaotic orbits and prevent them from exploring the entire $H = E$ manifold. But for $N \geq 3$ this is not so. It is possible for a single chaotic orbit to densely fill all of the "gaps" between tori. This wandering of the orbits is called "*Arnold Diffusion*".