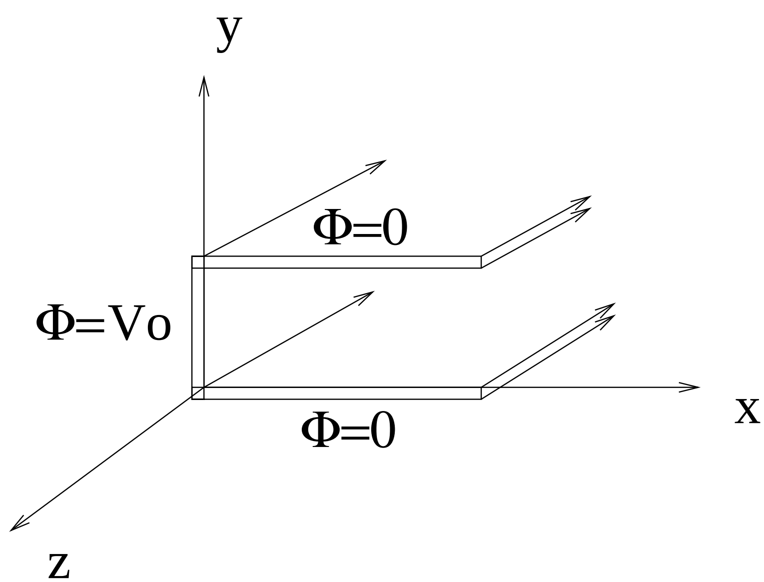
1 Topic 3: Separation of Variables and Series Expansions

Reading Assignment: Jackson Chapter 2.8 - 2.11

1.1 Separation of Variables

We will look at solving the Laplace equation by separation of variables. First, we'll just treat cartesian co-ordinates, and go through the subject with a series of examples.

1.1.1 Potential in an infinite "slot"



Consider two infinite metal plates lying parallel to the xz plane, one at $y = 0$, and the other at $y = \pi$. The left end at $x = 0$ is colosed off with an infinite strip insulated from the two plates, and maintained at potential $V_o(y)$. Find the potential everywhere in the slot.

Using a technique you will remember from the beginning of the term, we turn Laplace's *partial* differential equation into an *ordinary* differential equation by separating variables. We want to solve

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{1}$$

subject to:

$$\begin{aligned} \Phi &= 0 \text{ when } y = 0 \\ \Phi &= 0 \text{ when } y = \pi \\ \Phi &= V_o(y) \text{ when } x = 0 \\ \Phi &\to 0 \text{ as } x \to \infty \end{aligned}$$

Look for solutions of the form

$$\Phi(x,y) = X(x)\,Y(y) \tag{2}$$

Now this may seem like a restrictive condition for the solution, but as we'll see, the fact that solving this allows us to find a large number of specific solutions that can be superimposed to solve arbitrary problems (with rectangular symmetry).

Plugging in, we get

$$\begin{aligned} Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} &= 0 \\ \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} &= 0 \end{aligned}$$

since this is of the form

$$f(x) + g(y) = 0 \tag{3}$$

where f and g are functions of independent variables. Therefore, there's a solution only if f, g are constants:

$$\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= C_1 \\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= C_2 \end{aligned}$$

and

$$C_1 + C_2 = 0 \tag{4}$$

one of the constants is positive, the other must be negative. In general what we have to do is consider both possibilities, and choose the one appropriate for the problem at hand. For reasons that will be clear later, I'll choose C_1 positive, and C_2 negative

$$\frac{d^2X}{dx^2} = k^2X, \quad \frac{d^2Y}{dy^2} = -k^2Y \tag{5}$$

So, we have *converted a PDE to two ODEs*, which are a lot easier to solve. In fact, we can write down the solutions by inspection

$$\begin{aligned} X(x) &= Ae^{kx} + Be^{-kx} \\ Y(y) &= C\sin ky + D\cos ky \end{aligned}$$

so that

$$\Phi(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky) \tag{6}$$

We have an appropriate solution, now we just have to look at the boundary conditions. $\Phi \to 0$ as $x \to \infty$ require $A = 0$, so

$$\Phi(x,y) = e^{-kx}(C\sin ky + D\cos ky) \tag{7}$$

$\Phi = 0$ when $y = 0$ requires $D = 0$, so

$$\Phi(x, y) = Ce^{-kx} \sin ky \tag{8}$$

and $\Phi = 0$ when $y = \pi$ requires k to be an integer.

The question is, how do we fit the final boundary condition at $x = 0$ (unless $\Phi(0, y) = C \sin ky$)?

The key is that Laplace's equation is linear, in that if $\Phi_1, \Phi_2, ...$ are solutions, then so is $\Phi = \alpha_1 \Phi_1 + \alpha_2 \Phi_2 + ...$ We can construct a general solution by summing the infinite number of solutions we found

$$\Phi(x, y) = \sum_{k=1}^{\infty} C_k e^{-kx} \sin ky \tag{9}$$

Can we, by proper choice of $C_k' s$ guarantee that we can match the last boundary condition:

$$\Phi(0, y) = \sum_{k=1}^{\infty} C_k \sin ky = V_o(y) \tag{10}$$

You probably recognize this as a Fourier sine series. Virtually any function $V_o(y)$ can be expanded in such a series.

How do we actually evaluate the C's? If we multiply both sides by $\sin(ny)$ and integrate from $0, \pi$

$$\sum_{k=1}^{\infty} C_k \int_0^{\pi} \sin ky \ \sin ny \ dy = \begin{matrix} 0 \text{ if } k \neq n \\ \pi/2 \text{ if } k = n \end{matrix} \tag{11}$$

All terms in the series drop out except where $k = n$, and we have

$$C_n = \frac{2}{\pi} \int_0^{\pi} V_o(y) \sin ny \, dy \tag{12}$$

Suppose, for example, that the end is held at constant potential, then

$$C_n = \frac{2V_o}{\pi} \int_0^{\pi} \sin ny \, dy = \frac{2V_o}{n\pi} (1 - \cos n\pi) = \begin{matrix} 0 \text{ - } n \text{ even} \\ \frac{4V_o}{n\pi} \ \ n \text{ odd} \end{matrix} \tag{13}$$

so

$$\Phi(x, y) = \frac{4V_o}{\pi} \sum_{k \text{ odd}} \frac{1}{k} e^{-kx} \sin ky \tag{14}$$

The more terms we add, the closer we get to the actual solution. We can actually sum the infinite series explicitly, and get

$$\Phi(x, y) = \frac{2V_o}{\pi} \tan^{-1} \left(\frac{\sin y}{\sinh x} \right) \tag{15}$$

You can look at this form and see that all the boundary conditions are met, and check that Laplace's equation is obeyed.

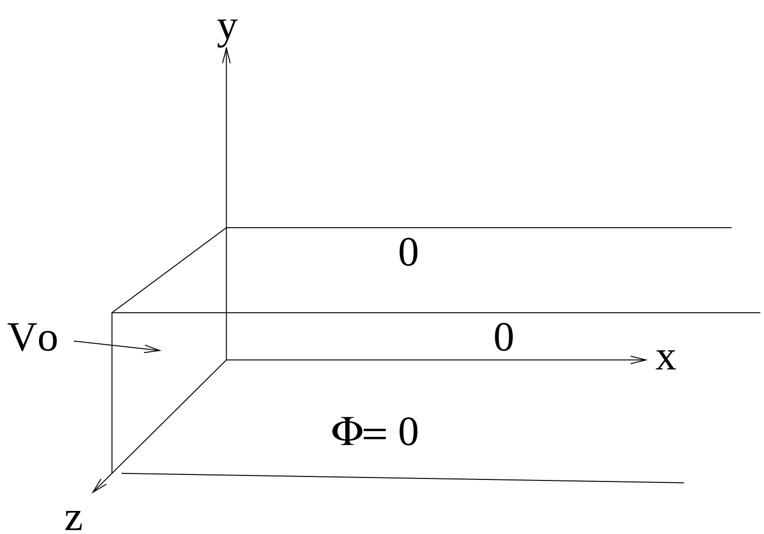
The fact that this method works in general is due to two important properties of the separable solutions, *completeness* and *orthogonality*. A set of functions $f_n(x)$ is complete if any other function, $f(x)$ can be expressed as a linear combination of them

$$f(x) = \sum_{n=1}^{\infty} C_n f_n(x) \tag{16}$$

The functions $\sin(nx)$ are complete on the interval $0 \leq x \leq \pi$. In general proving completeness is very difficult. The orthogonality property is expressed as

$$\int_a^b f_n(x) f_m(x)\, dx = 0 \qquad \text{for } n \neq m \tag{17}$$

The sines are orthogonal for the interval $a = 0 \to b = \pi$.



Lets look at a three-dimensional example: an infinitely long rectangular pipe with the sides grounded, but with one end $(x = 0)$ at $V_0(y, z)$ so we have

1) $\Phi = 0$ for $y = 0$
2) $\Phi = 0$ for $y = \pi$
3) $\Phi = 0$ for $z = 0$
4) $\Phi = 0$ for $z = \pi$
5) $\Phi \to 0$ as $x \to \infty$
6) $\Phi = V_o(y, z)$ for $x = 0$

Separate by looking for solutions where

$$\Phi(x, y, z) = X(x)\, Y(x)\, Z(z) \tag{18}$$

so substituting into LE we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{19}$$

and again

$$\begin{aligned}\frac{1}{X}\frac{d^2X}{dx^2} &= C_1\\ \frac{1}{Y}\frac{d^2Y}{dy^2} &= C_2\\ \frac{1}{Z}\frac{d^2Z}{dz^2} &= C_3\end{aligned}$$

with

$$C_1 + C_2 + C_3 = 0 \tag{20}$$

We want to guess which constants to choose as positive, and which as negative. Our previous experience suggests we want $C_{2,3}$ negative, and C_1 positive. Let $C_2 = -k^2, C_3 = -l^2, C_1 = k^2 + l^2$, so

$$\begin{aligned}\frac{d^2X}{dx^2} &= \left(k^2 + l^2\right) X\\ \frac{d^2Y}{dy^2} &= -k^2 Y\\ \frac{d^2Z}{dz^2} &= -l^2 Z\end{aligned}$$

so we have three *ordinary* differential equations. The solutions are

$$\begin{aligned}X(x) &= Ae^{\sqrt{k^2+l^2}x} + Be^{-\sqrt{k^2+l^2}x}\\ Y(y) &= C\sin ky + D\cos ky\\ Z(z) &= E\sin lz + F\cos lz\end{aligned}$$

From the boundary conditions we can argue that: $A = 0, D = 0, F = 0$, and k, l must be positive integers. So our solution is

$$\Phi(x,y,z) = Ce^{-\sqrt{k^2+l^2}x}\sin ky \sin lz \tag{21}$$

To satisfy the last boundary condition, we have to come up with a linear superposition of solutions, such that

$$\Phi(x,y,z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} C_{k,l}e^{-\sqrt{k^2+l^2}x}\sin ky \sin lz \tag{22}$$

We determine the coefficients so that the last boundary condition is satisfied:

$$\Phi(0,y,z) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} C_{k,l}\sin ky \sin lz = V_o(y,z) \tag{23}$$

using the orthogonality property,

$$C_{n,m} = \left(\frac{\pi}{2}\right)^2 \int_0^{\pi}\int_0^{\pi} V_o(y,z)\sin ny \sin mz \; dy\, dz \tag{24}$$

For example, if the potential at the endcap is constant, then

$$\begin{aligned} C_{n,m} &= \left(\frac{\pi}{2}\right)^2 V_o \int_0^\pi \int_0^\pi \sin ny \sin mz \; dy\, dz \\ &= \begin{array}{c} 0 \quad n \text{ or } \; m \text{ even} \\ \frac{16V_o}{\pi^2 nm} \quad n \text{ and } m \text{ odd} \end{array} \end{aligned}$$

and so

$$\Phi\left(x,y,z\right) = \frac{16V_o}{\pi^2} \sum_{n \text{ odd}} \sum_{m \text{ odd}} \frac{1}{nm} e^{-\sqrt{n^2+m^2}x} \sin ny \sin mz \tag{25}$$

Note that for x large, successive terms decrease rapidly, so we get a good approximation just by taking the first few.

1.2 Green's Functions

We're going to go back to the topic of Green's functions, which we investigated a bit last week. I mentioned that the method of images was one way to get Green's function for some situations – let's look at this in detail.

Remember that we started supposing that the potential $G\left(\mathbf{r},\mathbf{r}'\right)$ satisfies

$$\nabla_r^2 G\left(\mathbf{r};\mathbf{r}'\right) = -\frac{1}{\varepsilon_o}\delta(\mathbf{r}-\mathbf{r}') \tag{26}$$

for $\mathbf{r}'$ inside some volume V bounded by some surface S, and we imposed the special boundary condition that $G\left(S;\mathbf{r}'\right) = 0$ (by this notation I mean G disappears on the surface). This last condition is VERY IMPORTANT. We want to find $\Phi\left(\mathbf{r}\right)$ so that

$$\nabla^2\Phi\left(\mathbf{r}\right) = \frac{-1}{\varepsilon_o}\rho\left(\mathbf{r}\right) \tag{27}$$

for specified boundary conditions $\Phi\left(S\right)$.

Then Green's identity which we derived before is

$$\Phi\left(r'\right) = \int_V G\left(\mathbf{r};\mathbf{r}'\right)\rho\left(\mathbf{r}\right) d^3r - \varepsilon_o \int_S \Phi\left(\mathbf{r}\right)\nabla G\left(\mathbf{r};\mathbf{r}'\right)\cdot d\mathbf{a} \tag{28}$$

Because $\rho\left(\mathbf{r}\right)$ is given and $\Phi\left(S\right)$ is given, and by assumption we have solved $\nabla^2 G\left(\mathbf{r},\mathbf{r}'\right) = \frac{-1}{\varepsilon_o}\delta\left(\mathbf{r}-\mathbf{r}'\right)$ with $G\left(S;\mathbf{r}'\right) = 0$ we see that we have a practical formula for $\Phi\left(\mathbf{r}'\right)$. The problem is reduced to solving the problem of a unit point charge *at any point inside V with the boundary of V, S grounded* – i.e. set to zero potential.

The best way to understand this is to do some examples. First, though, we'll show the useful property that Green's function is symmetric;

$$G\left(\mathbf{r};\mathbf{r}'\right) = G\left(\mathbf{r}',\mathbf{r}\right) \tag{29}$$

with $G(S, \mathbf{r}') = 0$. So,

$$\begin{aligned} G(\mathbf{r}, \mathbf{t}) \nabla^2 G(\mathbf{r}, \mathbf{s}) &= \frac{-1}{\varepsilon_o} \delta(\mathbf{r} - \mathbf{s}) G(\mathbf{r}, \mathbf{t}) \\ G(\mathbf{r}, \mathbf{s}) \nabla^2 G(\mathbf{r}, \mathbf{t}) &= \frac{-1}{\varepsilon_o} \delta(\mathbf{r} - \mathbf{t}) G(\mathbf{r}, \mathbf{s}) \end{aligned}$$

and subtracting the two, integrating over the volume, and using Green's theorem,

$$\int_S [G(\mathbf{r}; \mathbf{t}) \nabla G(\mathbf{r}; \mathbf{s}) - G(\mathbf{r}; \mathbf{s}) \nabla G(\mathbf{r}; \mathbf{t})] \cdot da = -\frac{1}{\varepsilon_o} [G(\mathbf{s}, \mathbf{t}) - G(\mathbf{t}; \mathbf{s})] \tag{30}$$

The LHS is zero, since $G(S; \mathbf{t}) = G(S, \mathbf{s}) = 0$. Green's function is symmetric in its arguments.

We think of Green's theorem the following way:

$$\Phi(\mathbf{r}) = \int_V G(\mathbf{r}; \mathbf{r}') \rho(\mathbf{r}') d^3r' - \varepsilon_o \int_S \Phi(\mathbf{r}') \nabla' G(\mathbf{r}, \mathbf{r}') \cdot d\mathbf{a} \tag{31}$$

where the first part is the obvious part – each $\rho(\mathbf{r}') d^3r'$ produces $G(\mathbf{r}; \mathbf{r}')$ but the boundary conditions are not satisfied. The second part is not-obvious – it makes the boundary condion on S be satisfied.

I mentioned that the method of images was a way to find Green's function for some situations. Let's look at an example:

Examples – Images in a plane, Jackson 2.15 , and some more capacitance examples if we have time.

1.3 Laplace's Equation in Cylindrical Co-ordinates - corners and edges

We're going to look at separating Laplace's equations in cylindrical co-ordinates. Before tackling the general problem, we'll look at the specific case of an edge or corner, defined by two conducting planes. The angle between the planes can be $\beta < \pi/2$ (a corner) or $\beta > \pi/2$ (an edge). Our goal is to figure out what the charge density distribution will look like on the conducting planes near the corner or edge. To do this, we most naturally adopt cylindrical co-ordinates, as we consider points far away from the ends of the plane. In this case, we write Laplace's equation in cylindrical co-ordinates as

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\frac{\partial \Phi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{32}$$

Using the same separation of variable approach we did for Cartesian co-ordinates, we look for solutions of the form

$$\Phi(\rho, \phi) = R(\rho) \psi(\phi) \tag{33}$$

Plugging in above, and multiplying by $\rho^2 / (R\psi)$ we get

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\psi}\frac{d^2\psi}{d\phi^2}=0 \tag{34}$$

Using the similar argument that the part that is a function of ρ and the part that is a function of ψ are independent, and so must separately equal constants, we have

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)=k^2 \tag{35}$$

$$\frac{1}{\psi}\frac{d^2\psi}{d\phi^2}=-k^2 \tag{36}$$

The general solutions to these ODEs are

$$R(\rho)=a\rho^k+b\rho^{-k} \tag{37}$$

$$\psi(\phi)=A\cos(k\phi)+B\sin(k\phi) \tag{38}$$

where you'll note we chose judiciously when we decided which to set equal to the positive constant, and which the negative. Clearly we want the radial solutions to be the exponentials, so that they die out at $\rho\to\infty$ as they must, say, for an edge, and in ϕ we want sines and cosines, since we want to make boundaries defined by constant ϕ e.g. constant potential. Note that the case of $k=0$ is special. We can't stick $k=0$ in the solution above, but rather we have to look at the solution of the differential equations:

$$\begin{aligned}\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) &= 0\\ \frac{1}{\psi}\frac{d^2\psi}{d\phi^2} &= 0\end{aligned}$$

The solutions are obviously

$$\begin{aligned}R(\rho) &= a_o+b_o\ln\rho\\ \psi(\phi) &= A_o+B_o\phi\end{aligned}$$

If we imagine making ϕ single valued, then we have $k=n=$ positive or negative integer, and $B_o=0$. Our solutions are superpositions of the above basic functions:

$$\Phi(\rho)=a_o+b_o\ln\rho+\sum_{n=1}^{\infty}\left(a_n\rho^n+b_n\rho^{-n}\right)\left(A_n\cos(n\phi)+B_n\sin(n\phi)\right) \tag{39}$$

Now lets look at the particular problem of an edge or corner, where $0\le\phi\le\beta$, and where we fix the boundary condition as $\Phi=V_o$ for $\rho\ge0$, $\phi=0,\beta$. We require $b_o=b=0$ since our region of interest contains the origin, and the solution must be well-behaved there. For $\Phi=V_o$ at $\phi=0$ we must have $A_n=0$, and $\sin(k\beta)=0$. So $k=\frac{m\pi}{\beta}$ $m=1,2,....$. Finally, $a_0=V_o$, and

$$\Phi(\rho,\phi)=V_o+\sum_{m=1}^{\infty}a_m\rho^{m\pi/\beta}\sin(m\pi\phi/\beta) \tag{40}$$

We need one more condition to determine the constants a_m. This is the potential at large ρ. If we are very close to the edge or corner, then ρ is small, then successive terms decrease, and if we make ρ arbitrarily small, we need keep only the leading terms:

$$\Phi(\rho, \phi) = V_o + a_1 \rho^{\pi/\beta} \sin(\pi\phi/\beta) \tag{41}$$

We can get the electric field near the surface, and also the surface charge density from

$$\begin{aligned} E_\rho(\rho, \phi) &= -\frac{\partial \Phi}{\partial \rho} \approx \frac{-\pi a_1}{\beta} \rho^{\pi/\beta - 1} \sin(\pi\phi/\beta) \\ E_\phi(\rho, \phi) &= -\frac{1}{\rho}\frac{\partial \Phi}{\partial \phi} = -\frac{\pi a_1}{\beta} \rho^{\frac{\pi}{\beta} - 1} \cos(\pi\phi/\beta) \end{aligned}$$

The surface charge densities are

$$\sigma = \varepsilon_o E(\rho, 0) = \varepsilon_o E(\rho, \beta) = \frac{-\varepsilon_o \pi a_1}{\beta} \rho^{\pi/\beta - 1} \tag{42}$$

If we look at a corner, then $\beta < \pi$, and σ increases as ρ increases, and $\sigma \to 0$ as $\rho \to 0$. No charge accumulates at the corner. On the other hand, if we have an edge, $\beta > \pi$, and σ increases as ρ decreases, and charge accumulates at the point. If the point is infinitely sharp, the charge density becomes singular as $\rho \to 0$. If we have a thin sheet, $\beta = 2\pi$, and $\sigma \propto \rho^{-1/2}$– still integrable even though $\sigma \to \infty$ as $\rho \to 0$.

The high fields that occur near sharp edges is why lightening rods work. For some fixed potential between air and rod, charge accumulates at the tip, exceeding the breakdown field in air - causing an arc, or discharge originating at the tip – a way of controlling where lightening "hits". You can see the potentials are highest around the smallest radii of curvature if you consider two spheres with q_1, and q_2 and radii r_1 and r_2. If we connect them with a wire while they are infinitely far apart, the conductors are an equipotential, and from Gauss' law

$$\begin{aligned} E_1 &= q_1/\varepsilon_o \frac{1}{4\pi r_1^2} \\ E_2 &= q_2/\varepsilon_o \frac{1}{4\pi r_2^2} \end{aligned}$$

$$\frac{E_1}{E_2} = \frac{q_1}{q_2}\frac{r_2^2}{r_1^2} \tag{43}$$

and

$$\Phi_1 = q_1/\varepsilon_o \frac{1}{4\pi r_1} = q_2/\varepsilon_o \frac{1}{4\pi r_2} \tag{44}$$

so

$$\frac{E_1}{E_2} = \frac{r_2}{r_1} \tag{45}$$

and the electric field is highest for the smallest radius of curvature.